CHAPTER 3

Viscous and/or Heat Conducting Compressible Fluids

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HANDBOOK OF MATHEMATICAL FLUID DYNAMICS, VOLUME I
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1. Basic equations of mathematical fluid dynamics

1.1. *Balance laws*

Let $\Omega \subset R^N$ be a domain in two- or three-dimensional space ($N = 2, 3$) filled with a fluid. We shall assume the fluid is a continuous medium the *state* of which at a time $t \in I \subset R$ and a spatial point $x \in \Omega$ is characterized by the three fundamental macroscopic quantities; the *density* $\varrho = \varrho(t, x)$, the *velocity* $\mathbf{u} = \mathbf{u}(t, x)$, and the *temperature* $\theta = \theta(t, x)$. The fluid motion is governed by a system of partial differential equations expressing the basic principles of classical continuum mechanics.

Conservation of mass:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{u}) = 0 \tag{1.1}$$

Balance of momentum (*Newton's second law of motion*):

$$\frac{\partial(\varrho \mathbf{u})}{\partial t} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \Sigma + \varrho \mathbf{f} \tag{1.2}$$

Conservation of energy (*the first law of thermodynamics*):

$$\frac{\partial E}{\partial t} + \operatorname{div}\big((E + p)\mathbf{u}\big) = \operatorname{div}(\Sigma \mathbf{u}) - \operatorname{div} \mathbf{q} + \varrho \mathbf{f} \cdot \mathbf{u} \tag{1.3}$$

Here p is the *pressure*, Σ denotes the *viscous stress tensor*, E stands for the specific *energy*, $\mathbf{q}$ is the *heat flux*, and $\mathbf{f}$ denotes a given *external force* density.

We have chosen the *spatial description* where attention is focused on the present configuration of the fluid and the region of physical space currently occupied. This description was introduced by d'Alembert and is usually called *Eulerian* in hydrodynamics. There is an alternative way – the *referential description* – introduced in the eighteenth century by Euler that is called *Lagrangean*. In this description the Cartesian coordinate X of the position of the particle at the time $t = t_0$ is used as label for the particle X (see, e.g., Truesdell and Rajagopal [105]).

1.2. *Constitutive relations*

The general system (1.1)–(1.3) of $N + 2$ equations must be complemented by *constitutive relations* reflecting the diversity of materials in nature.

An important class of fluids that occupies a central place in fluid mechanics is the linearly viscous or *Newtonian fluid*, whose viscous stress tensor Σ takes the form

$$\Sigma = \Sigma(\nabla \mathbf{u}) = \mu\big(\nabla \mathbf{u} + (\nabla \mathbf{u})^t\big) + \lambda \operatorname{div} \mathbf{u} \operatorname{Id},$$

where μ and λ are the *viscosity coefficients* assumed to be constant unless otherwise specified.

The full stress characterized by the *Cauchy stress tensor* T is related to Σ by the *Stokes law*

$$\mathrm{T} = \Sigma - p\,\mathrm{Id},$$

where the pressure $p = p(\varrho, \theta)$ is a general function of the independent state variables ϱ and θ.

The specific energy E can be written in the form

$$E = E_{\mathrm{kinetic}} + E_{\mathrm{internal}}, \qquad E_{\mathrm{kinetic}} = \frac{1}{2}\varrho|\mathbf{u}|^2, \qquad E_{\mathrm{internal}} = \varrho e,$$

where e is the specific *internal energy* related to the density and the temperature by a general constitutive law $e = e(\varrho, \theta)$.

In accordance with the basic principles of thermodynamics, we postulate the existence of a new state variable – the *specific entropy* $S = S(\varrho, \theta)$ – satisfying

$$\frac{\partial S}{\partial \theta} = \frac{1}{\theta}\frac{\partial e}{\partial \theta}, \qquad \frac{\partial S}{\partial \varrho} = \frac{1}{\theta}\left(\frac{\partial e}{\partial \varrho} - \frac{p}{\varrho^2}\right). \tag{1.4}$$

Consequently, one can replace (1.3) by the entropy equation

$$\partial_t(\varrho S) + \operatorname{div}\left(\varrho S\mathbf{u} + \frac{\mathbf{q}}{\theta}\right) = \frac{\Sigma(\nabla\mathbf{u}) : \nabla\mathbf{u}}{\theta} - \frac{\mathbf{q}\cdot\nabla\theta}{\theta^2}, \tag{1.5}$$

where, by virtue of the second law of thermodynamics, the right-hand side should be non-negative which yields the restriction

$$\mu \geqslant 0, \qquad \lambda + \frac{2}{N}\mu \geqslant 0 \quad \text{together with } \mathbf{q}\cdot\nabla\theta \geqslant 0.$$

We focus on *viscous fluids* assuming always

$$\mu > 0, \qquad \lambda + \frac{2}{N}\mu \geqslant 0. \tag{1.6}$$

Finally, the heat flux $\mathbf{q}$ is related to the temperature by the *Fourier law*

$$\mathbf{q} = -\kappa\nabla\theta, \quad \kappa \geqslant 0,$$

where the *heat conduction coefficient* κ may depend on θ, ϱ and even on $\nabla\theta$ though it is assumed constant in most of the cases we shall deal with.

1.3. *Barotropic models*

The flow is said to be *barotropic* if the pressure p depends solely on the density ϱ. There are several situations when such a hypothesis seems appropriate. For instance, the *ideal gas* consitutive relation for the pressure reads

$$p = (\gamma - 1)\varrho e, \quad e = c_v \theta, \ c_v > 0, \tag{1.7}$$

where $\gamma > 1$ is the *adiabatic constant*. Accordingly, the entropy S takes the form

$$S = \log(e) + (1 - \gamma)\log(\varrho).$$

Substituting $\mu = \lambda = \kappa = 0$ in (1.5) and assuming a spatially homogeneous distribution S_0 of the entropy at a time $t_0 \in I$ we easily deduce $S(t) = S_0$ for any $t \in I$ and, consequently,

$$p(\varrho) = a\varrho^{\gamma}, \quad a = (\gamma - 1)\exp(S_0) > 0.$$

Under such circumstances, Equations (1.1), (1.2) represent a closed system describing the motion of an *isentropic compressible viscous fluid*.

A similar situation occurs in the *isothermal case* when we suppose $\theta(t) = \theta_0$ and (1.7) reduces to

$$p(\varrho) = r\theta_0\varrho, \quad r > 0.$$

For a general barotropic flow, the specific energy E can be taken in the form

$$E[\varrho, \mathbf{u}] = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \tag{1.8}$$

with

$$P'(z)z - P(z) = p(z).$$

The energy of a barotropic flow satisfies the equality

$$\partial_t E + \operatorname{div}\big((E + p)\mathbf{u}\big) = \operatorname{div}(\Sigma\mathbf{u}) - \Sigma : \nabla\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u} \tag{1.9}$$

which is now a direct consequence of (1.1), (1.2).

From the mathematical point of view, the barotropic flows represent an interesting class of problems for which an existence theory with basically no restriction on the size of data is available (see Section 6 below).

1.4. *Boundary conditions*

To obtain mathematically well-posed problems, the equations introduced above must be supplemented by initial and/or boundary conditions. The boundary $\partial\Omega$ is assumed to be an impermeable rigid wall, i.e., the fluid does not cross the boundary but may move tangentially to the boundary. Accordingly, we require

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } I \times \partial\Omega, \tag{1.10}$$

where $\mathbf{n}$ stands for the outer normal vector. For both experimental and mathematical reasons, (1.10) should be accompanied by a condition for the tangential component of the velocity. As observed in experiments with viscous fluids, the tangential component approaches zero at the boundary to a high degree of precision. This can be expressed by the *no-slip boundary conditions*:

$$\mathbf{u} = 0 \quad \text{on } I \times \partial\Omega. \tag{1.11}$$

On the other hand, in vessels with frictionless boundary (cf. Ebin [25]), condition (1.10) is usually complemented by the requirement that the tangetial component of the normal stress is zero, which can be written in the form of the *no-stick boundary conditions*:

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad (\Sigma \mathbf{n}) \times \mathbf{n} = 0 \quad \text{on } I \times \partial\Omega. \tag{1.12}$$

Similarly, one prescribes either the heat flux or the temperature. For a thermally insulated boundary, the condition reads

$$\mathbf{q} \cdot \mathbf{n} = 0 \quad \text{on } I \times \partial\Omega$$

while

$$\theta = \theta_{\mathrm{b}} \quad \text{on } I \times \partial\Omega$$

when the boundary distribution of the temperature is known.

If Ω is unbounded, it is customary to prescribe also the limit values of the state variables for large $x \in \Omega$, e.g.,

$$\varrho \to \varrho_\infty, \qquad \mathbf{u} \to \mathbf{u}_\infty, \qquad \theta \to \theta_\infty \quad \text{as } |x| \to \infty.$$

In the in-flow and/or out-flow problems, the homogeneous Dirichlet boundary conditions (1.11) are replaced by a more general stipulation

$$\mathbf{u} = \mathbf{u}_{\mathrm{b}} \quad \text{on } I \times \partial\Omega.$$

Moreover, the density distribution must be given on the in-flow part of the boundary, i.e.,

$$\varrho(t, x) = \varrho_{\mathrm{b}}(x) \quad \text{for } (t, x) \in I \times \partial\Omega, \qquad \mathbf{u}_{\mathrm{b}}(x) \cdot \mathbf{n}(x) < 0.$$

Other types of boundary conditions including unilateral constraints and free boundary problems are treated in the monograph by Antontsev et al. [4, Chapters 1, 3].

1.5. *Bibliographical comments*

An elementary introduction to the mathematical theory of fluid mechanics can be found in the book by Chorin and Marsden [12]. More extensive material is available in the monographs by Batchelor [7], Meyer [77], Serrin [93], or Shapiro [94]. A more recent treatment including the so-called alternative models is presented by Truesdell and Rajagopal [105].

A rigorous mathematical justification of various models of viscous heat conducting fluids is given by Šilhavý [96]. Mainly mathematical aspects of the problem are discussed by Antontsev et al. [4], Málek et al. [68], and more recently by Lions [61,62].

2. Mathematical aspects of the problem

The first and most important criterion of applicability of *any* mathematical model is its *well-posedness*. According to Hadamard, this issue comprises a thorough discussion of the following topics.

- *Existence of solutions for given data*. The *data* for the problem in question are usually the values of the state variables ϱ, $\mathbf{u}$, and θ specified at a given time $t = t_0$ and/or the driving force $\mathbf{f}$ together with the boundary values of certain quantities as the case may be. The problem is whether or not there exist solutions for any choice of the data on a given time interval I.
- *Uniqueness*. The model is to be deterministic, specifically, the time evolution of the system for $t > t_0$ must be uniquely determined by its state at the time t_0.
- *Stability*. Small perturbations of the data should result in small variation of the corresponding solution at least on a given compact time interval. On the other hand, experience with much simpler systems of ordinary differential equations suggests that chaotic behaviour may develop with growing time. Roughly speaking, the solutions may behave in a drastically different way in the long run no matter how close they might have been initially.

To begin, let us say honestly that a rigorous answer to most of the issues mentioned above is very far from being complete. Global existence and uniqueness of solutions to the system (1.1)–(1.3) is still a major open problem and only partial results shed some light on the amazing complexity of the problem.

In this introductory section, we review the presently available results on the existence of *classical* as well as *weak* or *distributional* solutions to the full system of equations of a compressible Newtonian and heat-conducting fluid. Equations (1.1)–(1.3) written in Cartesian coordinates take the form

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u^j)}{\partial x_j} = 0; \tag{2.1}$$

$$\frac{\partial(\varrho u^i)}{\partial t} + \frac{\partial(\varrho u^j u^i)}{\partial x_j} + \frac{\partial p}{\partial x_i}$$
$$= \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u^i}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left((\lambda + \mu)\left(\frac{\partial u^j}{\partial x_j}\right)\right) + \varrho f^i, \quad i = 1, \ldots, N; \tag{2.2}$$

$$c_v\left(\frac{\partial(\varrho\theta)}{\partial t}+\frac{\partial(\varrho\theta u^j)}{\partial x_j}\right)+p\left(\frac{\partial u^j}{\partial x_j}\right)$$
$$=\frac{\partial}{\partial x_j}\left(\kappa\frac{\partial\theta}{\partial x_j}\right)+\frac{\mu}{2}\left(\frac{\partial u^j}{\partial x_k}+\frac{\partial u^k}{\partial x_j}\right)^2+\lambda\left(\frac{\partial u^j}{\partial x_j}\right)^2. \tag{2.3}$$

Here and always in what follows, the summation convention is used.

The term classical solution means that the state variables have as many derivatives as necessary to give meaning to (2.1)–(2.3) on $\Omega\times I$ and are continuous up to the boundary $\partial\Omega$ to satisfy the boundary conditions as the case may be. Usually, we make no distinction between classical and *strong solutions* whose generalized derivatives are locally integrable functions and satisfy the equations almost everywhere in the sense of the Lebesgue measure. Typically, any strong solution is a classical one provided some additional smoothness of the data is assumed.

Multiplying (2.1)–(2.3) by a compactly supported and smooth *test function* φ and integrating the resulting expressions by parts, we get the integral identities:

$$\int_I\int_\Omega \varrho\frac{\partial\varphi}{\partial t}+\varrho u^j\frac{\partial\varphi}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t=0; \tag{2.4}$$

$$\int_I\int_\Omega \varrho u^i\frac{\partial\varphi}{\partial t}+\varrho u^i u^j\frac{\partial\varphi}{\partial x_j}+p\frac{\partial\varphi}{\partial x_i}-\mu\frac{\partial u^i}{\partial x_j}\frac{\partial\varphi}{\partial x_j}$$
$$-(\lambda+\mu)\frac{\partial u^j}{\partial x_j}\frac{\partial\varphi}{\partial x_i}+\varrho f^i\varphi\,\mathrm{d}x\,\mathrm{d}t=0,\quad i=1,\dots,N; \tag{2.5}$$
$$\int_I\int_\Omega c_v\left(\varrho\theta\frac{\partial\varphi}{\partial t}+\varrho\theta u^j\frac{\partial\varphi}{\partial x_j}\right)-p\left(\frac{\partial u^j}{\partial x_j}\right)\varphi-\kappa\frac{\partial\theta}{\partial x_j}\frac{\partial\varphi}{\partial x_j}$$
$$+\frac{\mu}{2}\left(\frac{\partial u^j}{\partial x_k}+\frac{\partial u^k}{\partial x_j}\right)^2\varphi+\lambda\left(\frac{\partial u^j}{\partial x_j}\right)^2\varphi\,\mathrm{d}x\,\mathrm{d}t=0. \tag{2.6}$$

We shall say that Equations (2.1)–(2.3) hold in $\mathcal{D}'(I\times\Omega)$ (in the sense of distributions) or, equivalently, that ϱ, $\mathbf{u}$, and θ is a *weak solution* of the problem if the integral identities (2.4)–(2.6) hold for any test function $\varphi\in\mathcal{D}(I\times\Omega)$. The symbol $\mathcal{D}(Q)$ denotes the space of infinitely differentiable functions with compact support in an open set Q.

It is not difficult to observe that the *local formulation* (2.1)–(2.3) and the *integral formulation* (2.4)–(2.6) are in fact *equivalent* provided the solution is smooth enough. On the other hand, (2.4)–(2.6) make sense under much weaker assumptions, namely, when the quantities ϱ, ϱu^i, $\varrho u^i u^j$, p, $\partial u_i/\partial x_j$, ϱf^i, $\varrho\theta$, $\varrho\theta u^j$, $p\partial u^j/\partial x_j$, $\partial\theta/\partial x_i$, $|\partial u^i/\partial x_j|^2$, $i,j=1,\dots,N$ are locally integrable on $I\times\Omega$. We shall always tacitly suppose that this is the case whenever speaking about weak solutions.

2.1. *Global existence for small and smooth data*

Following the pioneering work of Matsumura and Nishida [72] we consider the system (2.1)–(2.3) where

$$\mu=\mu(\varrho,\theta),\qquad \lambda=\lambda(\varrho,\theta)$$
$$\text{are smooth functions satisfying } \mu>0,\ \lambda+2/3\mu\geqslant 0; \tag{2.7}$$

the pressure p is given by a general constitutive relation

$$p=p(\varrho,\theta)\quad\text{and}\quad p,\ \frac{\partial p}{\partial\varrho},\ \frac{\partial p}{\partial\theta}>0; \tag{2.8}$$

$$\kappa=\kappa(\varrho,\theta),\quad \kappa>0. \tag{2.9}$$

In addition and in accordance with (1.4), we suppose

$$p(\varrho,\theta)=\frac{\partial p(\varrho,\theta)}{\partial\theta}\theta. \tag{2.10}$$

The problem (2.1)–(2.3) is complemented by the Dirichlet boundary conditions

$$u^i|_{\partial\Omega}=0,\quad i=1,2,3,\qquad \theta|_{\partial\Omega}=\theta_{\mathrm{b}}, \tag{2.11}$$

where $\theta_{\mathrm{b}}>0$; and the initial conditions

$$\begin{aligned}&\varrho(0,x)=\varrho_0(x)>0,\qquad u^i(0,x)=u^i_0(x),\quad i=1,2,3,\\&\theta(0,x)=\theta_0(x),\quad x\in\overline{\Omega}.\end{aligned} \tag{2.12}$$

The following global existence theorem holds.

THEOREM 2.1. *Let $\Omega\subset R^3$ be a domain with compact and smooth boundary. Let the quantities μ, λ, p, and κ comply with the hypotheses* (2.7)–(2.10). *Moreover, let the initial data ϱ_0, u^i_0, θ_0 belong to the Sobolev space $W^{3,2}(\Omega)$ and satisfy the compatibility conditions*

$$\begin{aligned}&u^i_0=0,\quad \theta_0=\theta_{\mathrm{b}},\\&\frac{\partial}{\partial x_i}p(\varrho_0,\theta_0)\\&\quad=\frac{\partial}{\partial x_j}\left(\mu(\varrho_0,\theta_0)\frac{\partial u^i_0}{\partial x_j}\right)+\frac{\partial}{\partial x_i}\left(\big(\lambda(\varrho_0,\theta_0)+\mu(\varrho_0,\theta_0)\big)\left(\frac{\partial u^j_0}{\partial x_j}\right)\right)-\varrho_0 f^i,\end{aligned}$$

$$
\begin{aligned}
&p(\varrho_0,\theta_0)\frac{\partial u_0^j}{\partial x_j}\\
&\quad=\frac{\partial}{\partial x_j}\left(\kappa(\varrho_0,\theta_0)\frac{\partial\theta_0}{\partial x_j}\right)+\frac{\mu(\varrho_0,\theta_0)}{2}\left(\frac{\partial u_0^j}{\partial x_k}+\frac{\partial u_0^k}{\partial x_j}\right)^2+\lambda(\varrho_0,\theta_0)\left(\frac{\partial u_0^j}{\partial x_j}\right)^2,\\
&\quad i=1,2,3,
\end{aligned}
$$

on the boundary $\partial\Omega$. *Finally, let* $f^i=\partial F/\partial x_i$, $i=1,2,3$ *where* F *belongs to the Sobolev space* $W^{5,2}(\Omega)$.

Then there exists $\varepsilon>0$ *such that the initial-boundary value problem* (2.1)–(2.3)*,* (2.11)*,* (2.12) *posseses a unique solution* ϱ, $\mathbf{u}$, θ *on the time interval* $t\in(0,\infty)$ *provided the initial data satisfy*

$$
\|\varrho_0-\bar{\varrho}\|_{W^{3,2}(\Omega)}+\|\mathbf{u}_0\|_{W^{3,2}(\Omega)}+\|\theta_0-\theta_b\|_{W^{3,2}(\Omega)}+\|F\|_{W^{5,2}(\Omega)}<\varepsilon,
$$

where

$$
\bar{\varrho}=\frac{1}{|\Omega|}\int_\Omega\varrho_0\,\mathrm{d}x.
$$

Theorem 2.1 in its present form is taken over from Matsumura and Nishida [73] (cf. also [72]). The proof, which is rather lengthy and technical, is based on a priori estimates resulting from energy relations. Their method has been subsequently adapted by many authors to attack various problems with non-homogeneous boundary conditions (cf., e.g., Valli and Zajaczkowski [109]) as well as the barotropic models (see Valli [108]). The common feature of all these results is that they apply only to problems where the data are small and regular.

The solutions the existence of which is claimed in Theorem 2.1 belong to the space

$$
\varrho,\mathbf{u},\theta\in C\big([0,T];W^{3,2}(\Omega)\big),
$$

where $W^{k,p}(\Omega)$ denotes the Sobolev space of functions whose derivatives up to order k lie in the Lebesgue space $L^p(\Omega)$ (for basic properties of Sobolev spaces see, e.g., the monograph of Adams [1]). The scale $W^{k,2}$ forms a suitable function spaces framework because of the variational structure of the problem. The *a priori* estimates are obtained in the Hilbertian scale $W^{k,2}$ in a very natural way via the "energy method" used in the pioneering paper by Matsumura [71]. Assuming more regularity of the initial data and F one could prove the same result with $W^{3,2}$ replaced by $W^{k,2}$ with k sufficiently large. It follows then from the standard *embedding* theorems that the solution would be classical. Alternatively, one can use the smoothing effect of the diffusion semigroup to deduce that the solutions constructed in Theorem 2.1 are, in fact, classical for $t>0$ (cf. Matsumura and Nishida [72]).

2.2. *Global existence of discontinuous solutions*

Discontinuous solutions are fundamental both in the physical theory of nonequilibrium thermodynamics and in the mathematical theory of models of inviscid fluids. It seems natural, therefore, to have a rigorous mathematical theory for the system (2.1)–(2.3) which would accommodate discontinuities in solutions. Of course, one has to abandon the classical concept of solution as a differentiable function and turn to the weak solutions which satisfy the integral identities (2.4)–(2.6).

It follows from (2.4)–(2.6) that the density ϱ, the momenta ϱu^i, $i = 1, 2, 3$, and the specific internal energy $\varrho\theta$ considered as vector functions of time are *weakly continuous*, i.e., the quantities

$$\int_\Omega \varrho\phi\,\mathrm{d}x, \qquad \int_\Omega \varrho u^i\phi\,\mathrm{d}x, \quad i = 1, \ldots, N, \qquad \int_\Omega \varrho\theta\phi\,\mathrm{d}x$$

belong to $C(I)$ for any fixed $\phi \in \mathcal{D}(\Omega)$.

Consequently, it makes sense to prescribe the initial conditions even in the class of weak solutions.

Pursuing this path Hoff [47] examined the system (2.1)–(2.3) on the whole space R^3 where the pressure p and the internal energy e obey the ideal gas constitutive relations (1.7) and the initial data ϱ_0, $\mathbf{u}_0$, θ_0 satisfy

$$\begin{aligned} &\varrho_0 - \bar{\varrho} \in L^\infty \cap L^2(R^3), \qquad \mathbf{u}_0 \in \left[W^{s,2}(R^3)\right]^3, \quad s \in (1/3, 1/2), \\ &\theta_0 - \bar{\theta} \in L^2(R^3), \end{aligned} \tag{2.13}$$

for certain positive constants $\bar{\varrho}$, $\bar{\theta}$. The Sobolev spaces $W^{s,2}(R^3)$ for a general real parameter s may be defined in terms of the Fourier transform (see Adams [1]).

We report the following result (see [47, Theorem 1.1]).

THEOREM 2.2. *Let $\Omega = R^3$. Assume that λ, μ and κ are constants satisfying*

$$\mu > 0, \qquad \mu/3 < \lambda + \mu < \left(1 + 1/3\sqrt{15}\right)\mu, \qquad \kappa > 0.$$

Let the pressure p obey the ideal gas constitutive relation

$$p = (\gamma - 1)c_{\mathrm{v}}\varrho\theta, \quad \gamma > 1,\ c_{\mathrm{v}} > 0, \tag{2.14}$$

and the initial data ϱ_0, $\mathbf{u}_0$, θ_0 satisfy (2.13) *for certain positive constants $\bar{\varrho}$, $\bar{\theta}$.*

Let positive constants $0 < \varrho_1 < \bar{\varrho} < \varrho_2$, $0 < \theta_2 < \theta_1 < \bar{\theta}$ be given. Finally, set $\mathbf{f} \equiv 0$ in (2.2).

Then there exists $\varepsilon > 0$ depending on ϱ_i, θ_i, $i = 1, 2$, and s such that the initial-value problem (2.1)–(2.3), (2.12) *possesses a weak solution ϱ, $\mathbf{u}$, θ on the set* $(0, \infty) \times R^3$

provided

$$\|\varrho_0 - \bar{\varrho}\|_{L^2 \cap L^\infty(R^3)} + \|\theta_0 - \bar{\theta}\|_{L^2(R^3)} + \|\mathbf{u}_0\|_{W^{s,2} \cap L^4(R^3)} < \varepsilon,$$
$$\operatorname{ess\,inf} e_0 \geqslant e_1.$$

Moreover, the solution satisfies

$$\varrho_1 \leqslant \varrho(t,x) \leqslant \varrho_2, \qquad \theta(t,x) \geqslant e_2 \quad \textit{for a.a. } (t,x) \in (0,\infty) \times R^3,$$

and

$$\varrho(t) \to \bar{\varrho}, \qquad \mathbf{u}(t) \to 0, \qquad \theta(t) \to \bar{\theta} \quad \textit{in } L^p\big(R^3\big) \textit{ as } t \to \infty$$

for any $2 < p \leqslant \infty$.

A similar result under slightly more restrictive hypotheses on the data can be proved for $\Omega = R^2$ (see Hoff [47, Theorem 1.1]).

The proof of Theorem 2.2 leans, among other things, on the regularity properties of the quantity $p - (\lambda + 2\mu) \operatorname{div} \mathbf{u}$ termed the *effective viscous flux*. More specifically, this quantity is shown to be free of jump discontinuities. This is the first indication of the important role played by the effective viscous flux in the mathematical theory of compressible fluids. We will address this issue in detail in Section 4.3.

2.3. *Global existence in critical spaces*

The solutions obtained in Theorem 2.2 solve the problem for a very general class of initial data but are not known to be unique. On the other hand, Theorem 2.1 yields a unique solution at the expense of higher regularity imposed on the data. A natural question to ask is how far one can get from the hypotheses of Theorem 2.1 to those of Theorem 2.2 to save uniqueness or, more precisely, what is a critical space of data for which the weak solutions are unique.

Such a question was already studied for the incompressible Navier–Stokes equations by Fujita and Kato [41]. The same problem for the full system (2.1)–(2.3) under rather general constitutive relations has been addressed only recently by Danchin [16]. He obtains existence and uniqueness of global solutions in a functional space setting invariant by the natural scaling of the associated equations:

$$\varrho_\nu(t,x) = \varrho\big(\nu^2 t, \nu x\big), \qquad \mathbf{u}_\nu = \nu \mathbf{u}\big(\nu^2 t, \nu x\big), \qquad \theta_\nu(t,x) = \nu^2 \theta\big(\nu^2 t, \nu x\big),$$

where the pressure law p is changed to $\nu^2 p$.

A functional space for the triple $[\varrho, \mathbf{u}, \theta]$ is termed *critical* if the associated norm is invariant under the transformation

$$[\varrho, \mathbf{u}, \theta] \mapsto [\varrho_\nu, \mathbf{u}_\nu, \theta_\nu]$$

up to a constant independent of ν. Accordingly, the well-posedness for the problem (2.1)–(2.3) can be stated in terms of the Besov spaces $B^s_{2,1}(R^3)$ whose exact definition goes beyond the framework of the present paper (see [16]). Let us only remark that the final result is of the same character as Theorem 2.1, namely, global existence and uniqueness of (weak) solutions of the problem (2.1)–(2.3) for data which are a small perturbation of a given equilibrium state.

2.4. *Regularity vs. blow-up*

Since the celebrated work of Leray, it has been a major open problem of mathematical fluid mechanics to prove or disprove that regular solutions of the *incompressible* Navier–Stokes equations in three space dimensions exist for all time. Clearly, the same problem for the general system (2.1)–(2.3) seems even more delicate.

As a matter of fact, there is a negative result of XIN [110, Theorem 1.3]. He considers the system (2.1)–(2.3) posed on the whole space R^3 with zero thermal conductivity $\kappa = 0$ and the initial density ϱ_0 compactly supported:

THEOREM 2.3. *Let $\Omega = R^3$ and $m > 3$ be a given number.*

Consider the system (2.1)–(2.3) *complemented by the initial conditions* (2.12) *where the viscosity coefficients λ, μ are constant and satisfy* (1.6)*, and p obeys the constitutive law* (2.14)*. Moreover, let $\kappa = 0$, $\mathbf{f} = 0$, and*

$$\varrho_0, \mathbf{u}_0, \theta_0 \in W^{m,2}\big(R^3\big),$$

$$\operatorname{supp} \varrho_0 \textit{ compact in } R^3, \qquad \theta_0 \geqslant \underline{\theta} > 0.$$

Then there is no solution of the initial value problem (2.1)–(2.3), (2.12) *such that*

$$\varrho, \mathbf{u}, \theta \in C^1\big([0,\infty); W^{m,2}\big(R^3\big)\big).$$

It seems interesting to compare the conclusion of Theorem 2.3 with the existence result of Theorem 2.1. Obviously, the above theorem does not seem to solve (in a negative way) the question of regularity for the compressible Navier–Stokes equations because of the hypothesis of compactness of the support of ϱ_0. We remark in this regard that the Navier–Stokes system is a model of nondilute fluids in which the density is bounded below away from zero. It is natural, therefore, to expect the problem to be ill-posed when vacuum regions are present at the initial time.

2.5. *Large data existence results*

To begin with, one should say there are practically no global existence results for the full system (2.1)–(2.3) when the data are allowed to be large. The question of local existence of classical solutions for regular initial data was addressed by Nash [79]. There is no indication, however, whether or not these solutions exist for all times.

Note that the problem here is of different nature than for systems of nonlinear conservation laws without diffusion terms. Indeed the equations (2.2), (2.3) are parabolic in $\mathbf{u}$, θ respectively provided the density ϱ is kept away from zero. Accordingly, one can anticipate these state variables to be regular provided uniform estimates of ϱ were available.

On the other hand, the density solves the hyperbolic equation (2.1) which is, however, only linear with respect to ϱ. Consequently, no shock waves should develop in ϱ provided they were not present initially and the velocity field $\mathbf{u}$ was sufficiently regular.

Formally, one can use the standard method of characteristics to deduce:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho\big(t, X(t)\big) + \varrho\big(t, X(t)\big)\operatorname{div}\mathbf{u}\big(t, X(t)\big) = 0,$$

where

$$X'(t) = \mathbf{u}\big(t, X(t)\big), \qquad X(0) = X_0 \in \Omega.$$

We end up in a "vicious circle" as we need uniform bounds on $\operatorname{div}\mathbf{u}$ to estimates the amplitude (and positivness) of ϱ but those are not available from the standard energy estimates. As indicated by Choe and Jin [11, Theorems 1.3, 1.4], the following three questions are intimately interrelated:

- uniform (on compact time intervals) upper bounds on the density ϱ;
- uniform boundedness below away from zero of ϱ;
- uniform bounds on $\mathbf{u}$.

Answering one of these questions would certainly lead to a rigorous large data existence theory in the framework of distributional (weak) solutions for the problem (2.1)–(2.3) (cf. also Lions [62]).

The above mentioned difficulties made several authors to search for a completely different approach to the problem. Motivated by the pioneering work of DiPerna [22], the theory of measure-valued solutions was developed by Málek et al. [68]. Roughly speaking, the "value" of each state variable at a fixed point (t, x) is no longer a number (or a finite component vector) but a probability measure (the Young measure) characterizing possible oscillations in a sequence of approximate solutions used to construct this particular variable. The numerical values of ϱ, $\mathbf{u}$, θ are centers of gravity of the corresponding Young measures and the nonlinear constitutive relations are expressed in a very simple way. These solutions are of course more general quantities than the distributional solutions and coincide with them provided one can show that the Young measures are concentrated at one point, i.e., they are Dirac masses for each value of the independent variables (t, x).

One can expect positive existence results in the class of measure valued solutions whenever suitable a priori estimates are available so that the nonlinear compositions are equi-integrable and consequently weakly compact in the space of Lebesgue integrable functions. This is of course considerably less than compactness of the state variables in the strong L^1-topology – an indispensable ingredient of any existence proof of distributional solutions.

The major shortcoming of measure-valued solutions is certainly the almost insurmountable problem of uniqueness solved only in the case of a scalar conservation law in [22].

This is, of course, the price to be paid for the relatively simple existence theory and one might feel tempted to say it is the same situation as when the weak solutions were introduced. However, this gap between existence and uniqueness, accepted for the weak solutions, seems to be simply too large in the class of measure-valued solutions and the approach is slowly being abandoned.

2.6. *Bibliographical comments*

Besides the results mentioned above, the small data existence problems were treated by Solonnikov [97], Tani [101], Valli [108] and others. The existence theory in critical spaces for barotropic flows was developed by Danchin [15]. As pointed out several times, the main obstacle to obtain large data existence results is the lack of suitable *a priori* estimates. Formal compactness results for the full system (1.1)–(1.3) were obtained by Lions [62] on condition of uniform boundedness of all state variables.

3. The continuity equation and renormalized solutions

Motivated by the work of Kruzkhov on scalar conservation laws, DiPerna and Lions [23] introduced the concept of renormalized solutions as a new class of solutions to general linear transport equations. They play a similar role as the entropy solutions in the theory of nonlinear conservation laws – they represent a class of physically relevant solutions in which the corresponding initial value problems admit a unique solution.

Multiplying (1.1) by $b'(\varrho)$, where b is a continuously differentiable function, we obtain the identity

$$\frac{\partial b(\varrho)}{\partial t} + \operatorname{div}\big(b(\varrho)\mathbf{u}\big) + \big(b'(\varrho)\varrho - b(\varrho)\big)\operatorname{div}\mathbf{u} = 0. \tag{3.1}$$

Obviously, any strong (classical) solution of (1.1) satisfies automatically (3.1). For the weak solutions, however, (3.1) represents an additional constraint which may not be always satisfied. Following [23] we shall say that ϱ is a *renormalized solution* of (1.1) on the set $I \times \Omega$ if ϱ, $\mathbf{u}$, $\nabla\mathbf{u}$ are locally integrable and (3.1) is satified in the sense of distributions (in $\mathcal{D}'(I \times \Omega)$), i.e., the integral identity

$$\int_I \int_\Omega b(\varrho)\frac{\partial \varphi}{\partial t} + b(\varrho)\mathbf{u}\cdot\nabla\varphi + \big(b(\varrho) - b'(\varrho)\varrho\big)\operatorname{div}\mathbf{u}\varphi \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.2}$$

holds for any test function $\varphi \in \mathcal{D}(I \times \Omega)$ and any $b \in C^1(R)$ such that

$$b'(z) = 0 \quad \text{for all } z \text{ large enough, say, } |z| \geqslant M. \tag{3.3}$$

Let us emphasize here that unlike the entropy solutions that can be characterized as satisfying a certain type of admissible jump conditions on discontinuity curves, the renormalized solutions characterize the so-called concentration phenomena (cf. Section 3.2 below).

3.1. *On continuity of the renormalized solutions*

The renormalized solutions enjoy many remarkable properties most of which can be proved by means of the regularization technique developed by DiPerna and Lions [23]. The following auxilliary assertion is classical (cf. Lions [61, Lemma 3.2]).

LEMMA 3.1. *Let $\Omega \subset R^N$ be a domain and*

$$\mathbf{u}, \nabla\mathbf{u} \in L^p_{\mathrm{loc}}(\Omega), \qquad \sigma \in L^q_{\mathrm{loc}}(\Omega),$$

where

$$1 \leqslant p, q \leqslant \infty, \qquad 1/r = 1/p + 1/q \leqslant 1.$$

Let ϑ_ε be a regularizing sequence, i.e., $\vartheta_\varepsilon \in \mathcal{D}(R^N)$, ϑ_ε radially symmetric and radially decreasing,

$$\int_{R^N} \vartheta_\varepsilon \,\mathrm{d}x = 1, \quad \vartheta_\varepsilon(x) \to 0 \text{ as } \varepsilon \to 0 \text{ for any fixed } x \in R^N \setminus \{0\}.$$

Then

$$\big\|\vartheta_\varepsilon * \big[\mathrm{div}(\sigma\mathbf{u})\big] - \mathrm{div}\big([\vartheta_\varepsilon * \sigma]\mathbf{u}\big)\big\|_{L^r(K)} \leqslant c(K)\|\mathbf{u}\|_{W^{1,p}(K)}\|\sigma\|_{L^q(K)}$$

for any compact $K \subset \Omega$ and

$$r_\varepsilon = \vartheta_\varepsilon * \big[\mathrm{div}(\sigma\mathbf{u})\big] - \mathrm{div}\big([\vartheta_\varepsilon * \sigma]\mathbf{u}\big) \to 0 \quad \text{in } L^r_{\mathrm{loc}}(\Omega) \text{ as } \varepsilon \to 0,$$

where $*$ stands for convolution on R^N.

Now one can regularize (3.1), more precisely, take $\phi(x) = \vartheta_\varepsilon(x - y)$ in (3.2) to deduce

$$\frac{\partial \vartheta_\varepsilon * b(\varrho)}{\partial t} + \mathrm{div}\big([\vartheta_\varepsilon * b(\varrho)]\mathbf{u}\big) + \vartheta_\varepsilon * \big[\big(b'(\varrho)\varrho - b(\varrho)\big)\,\mathrm{div}\,\mathbf{u}\big] = r_\varepsilon \tag{3.4}$$

for $t \in I$ and $x \in \Omega$ such that $\mathrm{dist}[x, \partial\Omega] > \varepsilon$. Here b is an arbitrary function satisfying (3.3) and $r_\varepsilon(t)$ as in Lemma 3.1, i.e., $r_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(I \times \Omega)$ as $\varepsilon \to 0$ provided $\mathbf{u}$ is locally integrable.

The first consequence of (3.4) is continuity in time of the renormalized solutions.

PROPOSITION 3.1. *Let $\mathbf{u}$, $\nabla\mathbf{u}$ be locally integrable on $I \times \Omega$ where $I \subset R$ is an open time interval and $\Omega \subset R^N$ a domain. Let ϱ – a locally integrable function – be a renormalized solution of the continuity equation* (2.1) *on $I \times \Omega$.*

Then for any compact $B \subset \Omega$ and any function b as in (3.3)*, the composition $b(\varrho)$: $t \in I \mapsto b(\varrho)(t)$ is a continuous function of t with values in the Lebesgue space $L^1(K)$, i.e.,*

$$b(\varrho) \in C\big(J; L^1(B)\big) \quad \text{for any compact } J \subset I.$$

Moreover, we have the following corollary.

COROLLARY 3.1. *In addition to the hypotheses of Proposition* 3.1*, assume that $I \subset R$, $\Omega \subset R^N$ are bounded; and*

$$\varrho \in L^\infty\big(0, T; L^p(\Omega)\big) \quad \text{for a certain } p > 1,$$

$$\mathbf{u}, \nabla u \in L^1(I \times \Omega).$$

Then ϱ as a function of $t \in I$ is continuous with values in $L^1(\Omega)$:

$$\varrho \in C\big(\bar{I}; L^1(\Omega)\big).$$

The proof of both Proposition 3.1 and Corollary 3.1 can be done via the regularization technique as in [23].

3.2. *Renormalized and weak solutions*

Another conclusion which can be deduced from Lemma 3.1 and (3.4) is that the class of weak and renormalized solutions coincide provided ϱ or $\nabla\mathbf{u}$ or both are sufficiently integrable.

PROPOSITION 3.2. *Assume*

$$\varrho \in L^p_{\mathrm{loc}}(I \times \Omega), \quad \mathbf{u}, \nabla\mathbf{u} \in L^q_{\mathrm{loc}}(I \times \Omega),$$

where

$$1 \leqslant p, q \leqslant \infty, \qquad 1/p + 1/q \leqslant 1.$$

Then ϱ is a renormalized solution of (2.1) *if and only if ϱ satisfies* (2.1) *in $\mathcal{D}'(I \times \Omega)$, i.e., the integral identity* (2.4) *holds for any test function $\varphi \in \mathcal{D}(I \times \Omega)$.*

Integrating (1.9) we can see that the typical regularity class for the velocity gradient is $\nabla\mathbf{u} \in L^2_{\mathrm{loc}}(I \times \Omega)$. Accordingly, to apply Proposition 3.2, one needs $\varrho \in L^2_{\mathrm{loc}}(I \times \Omega)$.

There is another reason why the density "should be" square integrable. The continuity equation can be (formally) written in the form

$$D_t\varrho + \varrho \operatorname{div}\mathbf{u} = 0, \quad \text{where } D_t = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\varrho$$

is the so-called *material derivative*. The quantity $\varrho \operatorname{div} \mathbf{u}$ plays the role of a forcing term in the above equation. Thus if we want to keep, at least in a certain weak sense, the structure given by characteristics (cf. Section 2.5), we should have $\varrho \operatorname{div} \mathbf{u}$ locally integrable. Taking the square integrability of $\nabla \mathbf{u}$ for granted we are led to require $\varrho \in L^2_{\mathrm{loc}}$ (cf. also DiPerna and Lions [23]). However, as we will see in Proposition 4.1 below, the square integrabilty of the density is not necessary for a weak solution of (1.1) to be a renormalized one.

3.3. *Renormalized solutions on domains with boundary*

Assume that $\Omega \subset R^N$ is a domain with Lipschitz boundary. As for the velocity field $\mathbf{u}$, we suppose $\mathbf{u} \in L^q_{\mathrm{loc}}(I \times \overline{\Omega})$, $\nabla \mathbf{u} \in L^q_{\mathrm{loc}}(I \times \overline{\Omega})$ for a certain $q > 1$. Although $\mathbf{u}$ need not be continuous, one can still consider the no-slip boundary conditions (1.11) in the sense of traces. Accordingly, assuming (1.11) and extending $\mathbf{u}$ to be zero outside Ω, one has $\mathbf{u} \in W^{1,q}_{\mathrm{loc}}(R^N)$. Equivalently, by virtue of the Hardy inequality (see, e.g., Opic and Kufner [85]), one can replace (1.11) by the following stipulation:

$$\frac{|\mathbf{u}|}{\operatorname{dist}[x, \partial\Omega]} \in L^q_{\mathrm{loc}}\big(I \times \overline{\Omega}\big). \tag{3.5}$$

Using (3.5) we can show a continuation theorem for renormalized solutions.

PROPOSITION 3.3. *Let $\Omega \subset R^N$ be a Lipschitz domain and let $\mathbf{u}$, $\nabla \mathbf{u}$ belong to $L^q_{\mathrm{loc}}(I \times \overline{\Omega})$ for a certain $q > 1$, and let* (1.11) *be satisfied in the sense of traces. Let ϱ be a renormalized solution of* (2.1) *on $I \times \Omega$.*

Then ϱ is a renormalized solution of (2.1) *on $I \times R^N$ provided ϱ, $\mathbf{u}$ are extended to be zero outside Ω.*

Proposition 3.3 together with Propositions 3.1, 3.2 yield an interesting corollary, namely, the *principle of total mass conservation* for the weak solutions of (2.1). Consider a bounded domain $\Omega \subset R^N$ with Lipschitz boundary on which $\mathbf{u}$ satisfies the no-slip boundary condition (1.11). Formally, one can integrate (2.1) over Ω to deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho \, \mathrm{d}x = 0,$$

i.e., the *total mass*

$$m = \int_\Omega \varrho \, \mathrm{d}x$$

is a constant of motion. By virtue of Propositions 3.1–3.3, we have the same result for distributional solutions:

PROPOSITION 3.4. *Let $\Omega \subset R^N$ be a bounded Lipschitz domain. Let*

$$\varrho \in L^\infty\big(I; L^p(\Omega)\big), \qquad \mathbf{u}, \nabla\mathbf{u} \in L^q(I \times \Omega),$$
$$1 < p, q \leqslant \infty,\ 1/p + 1/q \leqslant 1,$$

solve (2.1) *in $\mathcal{D}'(I \times \Omega)$ and*

$$\mathbf{u}|_{\partial\Omega} = 0.$$

Then the total mass

$$m = \int_\Omega \varrho(t)\,\mathrm{d}x$$

is constant for $t \in I$.

4. Weak convergence results

Many of the most important techniques set forth in recent years for studying the problem (2.1)–(2.3) are based on weak convergence methods. To establish the existence of a solution, an obvious idea is first to invent an appropriate collection of approximating problems, which can be solved; and then to pass to the limit in the sequence of approximate solutions to obtain a solution of the original problem. The overall impediment of this approach is of course the nonlinearity. Whereas it is very often true that one can find certain uniform estimates on the family of approximate solutions, the bounds on oscillations of these quantites are usually in short supply. This is, for instance, the case of the density ϱ solving the hyperbolic equation (2.1).

In this section, we shall investigate the compactness properties of weakly convergent sequences of solutions of the continuity equation (2.1) and the momentum equations (2.2). More precisely, we consider a family of weak solutions $\{\varrho_n\}$, $\{\mathbf{u}_n\}$ of the system (2.1), (2.2), i.e., the integral identities

$$\int_I\int_\Omega \varrho_n \frac{\partial\varphi}{\partial t} + \varrho_n u_n^j \frac{\partial\varphi}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{4.1}$$

$$\begin{aligned}\int_I\int_\Omega &\varrho_n u_n^i \frac{\partial\varphi}{\partial t} + \varrho_n u_n^i u_n^j \frac{\partial\varphi}{\partial x_j} + p_n \frac{\partial\varphi}{\partial x_i} - \mu \frac{\partial u_n^i}{\partial x_j}\frac{\partial\varphi}{\partial x_j}\\ &- (\lambda+\mu)\frac{\partial u_n^j}{\partial x_j}\frac{\partial\varphi}{\partial x_i} + \varrho f_n^i \varphi\,\mathrm{d}x\,\mathrm{d}t = 0,\\ &i = 1,\dots,N,\ n = 1,2,\dots,\end{aligned} \tag{4.2}$$

hold for any test function $\varphi \in \mathcal{D}(I \times \Omega)$. Since all results we shall discuss are of local nature, we assume that both the time interval I and the spatial domain Ω are bounded.

Moreover, we suppose that ϱ_n, $\mathbf{u}_n$, $\nabla\mathbf{u}_n$, p_n, and $\mathbf{f}_n$ are locally integrable and weakly convergent, specifically,

$$\left.\begin{aligned} \varrho_n &\to \varrho \\ \mathbf{u}_n &\to \mathbf{u} \\ \nabla\mathbf{u}_n &\to \nabla\mathbf{u} \\ p_n &\to p \\ \mathbf{f}_n &\to \mathbf{f} \end{aligned}\right\} \quad \text{weakly } \bigl(\text{in } \mathcal{D}'(I \times \Omega)\bigr) \text{ as } n \to \infty.$$

Here $v_n \to v$ *weakly* means that v is locally inegrable on $I \times \Omega$ and

$$\int_I \int_\Omega v_n \varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_I \int_\Omega v\varphi \,\mathrm{d}x\,\mathrm{d}t \quad \text{for any } \varphi \in \mathcal{D}(I \times \Omega).$$

Our goal in this section is to identify the limit problem solved by the quantities ϱ, $\mathbf{u}$, p, and $\mathbf{f}$. The best possible result is, of course, they satisfy the same system of equations. If this is the case, the problem enjoys the *property of compactness* with respect to the weak topology. Given relatively feeble *a priori* estimates (cf. Section 5), the weak compactness of the problem plays a decisive role in the larga data existence theory for barotropic flows presented in Section 6.

4.1. *Weak compactness of bounded solutions to the continuity equation*

Although hyperbolic, the continuity equation exhibits the best properties as far as the weak compactness of solutions is concerned.

Consider a sequence ϱ_n, $\mathbf{u}_n$ of renormalized solutions of (1.1) on $I \times \Omega$, i.e., in addition to (4.1), we assume (3.2) holds for any b as in (3.3). Moreover, we shall assume that

$$\|\mathbf{u}_n\|_{L^1(I\times\Omega)},\ \|\nabla\mathbf{u}_n\|_{L^q(I\times\Omega)} \leqslant c \quad \text{for a certain } q > 1; \tag{4.3}$$

and that the family ϱ_n is equi-bounded and equi-integrable, i.e.,

$$\begin{aligned} &\int_I \int_\Omega \varrho_n \,\mathrm{d}x\,\mathrm{d}t \leqslant c, \\ &\lim_{|Q|\to 0} \int_Q \varrho_n \,\mathrm{d}x\,\mathrm{d}t = 0 \quad \text{uniformly with respect to } n = 1, 2, \ldots. \end{aligned} \tag{4.4}$$

The failure of weak convergence to imply strong convergence is usually recorded by certain measures called *defect measures*. To this end, we introduce the cut-off operators $T_k = T_k(z)$,

$$T_k(z) = kT\left(\frac{z}{k}\right), \quad k \geqslant 1, \tag{4.5}$$

where $T \in C^1(R)$ is an odd function such that

$$T(z) = z \quad \text{for } 0 \leqslant z \leqslant 1, \qquad T(z) = 2 \quad \text{for } z \geqslant 3, \qquad T \text{ concave on } [0, \infty).$$

The amplitude of possible oscillations in the density sequence will be measured by the quantity

$$\operatorname{osc}_p[\varrho_n - \varrho](Q) = \sup_{k \geqslant 1}\Big(\limsup_{n\to\infty} \|T_k(\varrho_n) - T_k(\varrho)\|_{L^p(Q)}\Big).$$

Unlike the defect measures introduced by DiPerna and Majda [24], osc vanishes on any set on which ϱ_n tends to ϱ strongly in the L^1-topology regardless possible concentration effects.

Now we shall address the following question: Under which conditions do the limit functions ϱ, $\mathbf{u}$ solve (2.1)? Taking a function b as in (3.3) one deduces easily from (3.2), (4.3), and (4.4) that

$$b(\varrho_n) \to \overline{b(\varrho)} \quad \text{in } C\big(I; L^p_{\text{weak}}(\Omega)\big),\ 1 \leqslant p < \infty. \tag{4.6}$$

Here and in what follows, we shall use the standard notation $\overline{g(v)}$ for a weak (L^p) limit of a sequence $g(v_n)$ where v_n tends weakly to v. The possibility to find a subsequence of v_n such that the composition $b(v_n)$ is weakly convergent for *any* continuous b satisfying certain growth conditions is the basic statement of the theory of Young measures (cf. Tartar [102,103], Pedregal [88]). Such a limit, however, need not be unique unless the convergence of v_n is strong.

A sequence v_n converges to v in $C(I; L^p_{\text{weak}}(\Omega))$ if it is bounded in $L^\infty(I; L^p(\Omega))$, the function $t \mapsto \int_\Omega v_n(t,x)\phi(x)\,\mathrm{d}x$ can be identified with a continuous function on I, and

$$\int_\Omega v_n(t,x)\phi(x)\,\mathrm{d}x \to \int_\Omega v(t,x)\phi(x)\,\mathrm{d}x \quad \text{uniformly with respect to } t \in I$$

for any test function $\phi \in \mathcal{D}(\Omega)$.

By virtue of the Aubin–Lions lemma (see, e.g., Lions [59, Theorem 5.1], the relations (4.3), (4.6) imply

$$b(\varrho_n)\mathbf{u}_n \to \overline{b(\varrho)}\mathbf{u} \quad \text{weakly in } L^q_{\text{loc}}(I \times \Omega). \tag{4.7}$$

Indeed taking p large enough in (4.6) we get $L^p(\Omega)$ compactly imbedded in $W^{-1,q}(\Omega)$; whence

$$b(\varrho_n) \to \overline{b(\varrho)} \quad \text{in } C\big(I; W^{-1,q}(\Omega)\big)$$

which together with (4.3) yields the desired conclusion. The distributions lying in the "negative" Sobolev space $W^{-1,q}$ can be identified with generalized derivatives of vector functions in L^q (see, e.g., Adams [1]). In particular, if we knew that ϱ_n is bounded in $L^p_{\text{loc}}(I \times \Omega)$, we could conclude that ϱ, $\mathbf{u}$ solve (2.1) in the sense of distributions.

In a general case, we report the following result (cf. [29, Proposition 7.1]):

PROPOSITION 4.1. *In addition to* (4.1), *assume* ϱ_n, $\mathbf{u}_n$ *satisfy* (2.1) *in the sense of renormalized solutions on* $I \times \Omega$. *Moreover let* (4.3), (4.4) *hold, and*

$$\operatorname{osc}_p[\varrho_n - \varrho](Q) \leqslant c(Q) \quad \textit{for any compact } Q \subset I \times \Omega,$$

where

$$\frac{1}{p} + \frac{1}{q} < 1.$$

Then ϱ, $\mathbf{u}$ *is a renormalized solution of* (2.1) *on* $I \times \Omega$.

The main advantage of Proposition 4.1 is that the sequence ϱ_n itself need not be bounded in L^p. As we will see later, this is particularly convenient when barotropic fluids are studied (cf. Proposition 5.3 below).

4.2. *On compactness of solutions to the equations of motion*

In this part, we shall assume that

$$\|\mathbf{u}_n\|_{L^2(I\times\Omega)},\ \|\nabla\mathbf{u}_n\|_{L^2(I\times\Omega)} \leqslant c \quad \text{for all } n = 1, 2, \ldots. \tag{4.8}$$

In particular, the products $u^i u^j$, $i, j = 1, \ldots, N$, are bounded in $L^1(I, L^{2^*/2}(\Omega))$ where 2^* is the Sobolev exponent for the embedding $W^{1,2} \subset L^{2^*}$ to hold, i.e.,

$$2^* \text{ is arbitrary finite for } N = 2 \quad \text{and} \quad 2^* = \frac{2N}{N-2} \quad \text{if } N = 3, 4, \ldots.$$

Consequently, for the cubic quantity $\varrho_n u_n^i u_n^j$ to be at least integrable, it is neccessary that

$$\operatorname*{ess\,sup}_{t\in I} \big\|\varrho_n(t)\big\|_{L^p(\Omega)} \leqslant c \quad \text{for a certain } p \geqslant N/2 \tag{4.9}$$

provided $N > 2$. Here again, we face one of the major obstacles to build up a rigorous mathematical theory for the full system (2.1)–(2.3), namely, the lack of suitable a priori estimates. The only available bounds on the density are those deduced from boundedness of the total energy. In general, these "energy" estimates are not sufficient to get (4.9). Of course, the barotropic case offers a considerable improvement as the energy is given by formula (1.8) and, consequently, the desired estimates follow provided

$$\gamma \geqslant N/2.$$

The situation is more delicate in the physically relevant two-dimensional case. Here, the Sobolev space $W^{1,2}$ is embedded in the Orlicz space L^Φ generated be the function

$$\Phi(z) = \exp\big(z^2\big) - 1$$

(see Adams [1]). Consequently, condition (4.9) should be replaced by

$$\operatorname*{ess\,sup}_{t\in I}\int_\Omega \Psi(\varrho_n)\,\mathrm{d}x \leqslant c \quad \text{with } \Psi(z) \geqslant z\log(z). \tag{4.10}$$

Since ϱ_n, $\mathbf{u}_n$ satisfy also the continuity equation (4.1) we deduce from (4.9), (4.10) respectively that

$$\varrho_n \to \varrho \quad \text{in } C\big(I; L^p_{\text{weak}}(\Omega)\big),\ p \geqslant N/2 \text{ if } N = 3, \ldots, \tag{4.11}$$

or

$$\varrho_n \to \varrho \quad \text{in } C\big(I; L^\Psi_{\text{weak}}(\Omega)\big) \text{ for } N = 2. \tag{4.12}$$

In both cases this implies compactness of ϱ_n in $L^2(I, W^{-1,2}(\Omega))$, and we get

$$\varrho_n\mathbf{u}_n \to \varrho\mathbf{u} \quad \text{weakly in, say, } L^1(I\times\Omega).$$

The weak compactness of the cubic term $\varrho_n u_n^i u_n^j$ represents a more difficult problem. In addition to the above hypotheses, we assume the kinetic energy to be bounded uniformly in n, i.e.,

$$\varrho_n|\mathbf{u}_n|^2 \text{ is bounded in } L^1(I\times\Omega) \text{ uniformly with respect to } n.$$

Supposing (4.9) holds for $p > N/2$ we have, similarly as above,

$$\varrho_n\mathbf{u}_n \to \varrho\mathbf{u} \quad \text{in } C\big(I; L^r_{\text{weak}}(\Omega)\big) \text{ for a certain } r > \frac{2N}{N+2},\ N \geqslant 2; \tag{4.13}$$

whence

$$\varrho_n u_n^i u_n^j \to \varrho u^i u^j \quad \text{weakly in, say, } L^1(I\times\Omega),\ i,j = 1,\ldots,N,\ N\geqslant 2.$$

As a matter of fact, the result is not optimal for $N = 2$; in that case one could use directly (4.12) provided Ψ was a function dominating $z\log(z)$.

Summing up the previous considerations we get the following conclusion:

PROPOSITION 4.2. *Let the quantities* ϱ_n, $\mathbf{u}_n$ *satisfy the estimates* (4.8), (4.9) *with* $p > N/2$. *Moreover, let the kinetic energy be bounded, specifically,*

$$\operatorname*{ess\,sup}_{t\in I}\int_\Omega \varrho_n(t)\big|\mathbf{u}_n(t)\big|^2\,\mathrm{d}x \leqslant c \quad \textit{for all } n = 1, 2, \ldots. \tag{4.14}$$

Finally, assume $\mathbf{f}_n$ *are bounded and*

$$\mathbf{f}_n \to \mathbf{f} \quad \textit{uniformly on } I\times\Omega. \tag{4.15}$$

Then the limit functions ϱ, $\mathbf{u}$, p, *and* $\mathbf{f}$ *satisfy* (2.1), (2.2) *in* $\mathcal{D}'(I \times \Omega)$, *i.e., the integral identities* (2.4), (2.5) *hold for any test function* $\varphi \in \mathcal{D}(I \times \Omega)$.

Let us repeat once more that it is an open problem whether or not the bounds required for the density component are really available. As we have seen in Section 2.5, uniform bounds on the density are equivalent to uniform boundedness of $\mathbf{u}$ – a situation reminiscent of the classical regularity problem for the incompressible Navier–Stokes equations.

4.3. *On the effective viscous flux and its properties*

Consider the quantity $p-(\lambda+2\mu)\operatorname{div}\mathbf{u}$ called usually the effective viscous flux. Formally, assuming all the functions in (2.2) smooth and vanishing for $|x|\to\infty$ we can compute

$$\begin{aligned} &p_n-(\lambda+2\mu)\operatorname{div}\mathbf{u}_n\\ &\quad=\Delta^{-1}\operatorname{div}(\varrho_n\mathbf{f}_n)-\Delta^{-1}\operatorname{div}(\varrho_n\mathbf{u}_n)_t-\mathcal{R}_{i,j}\big[\varrho_n u_n^i u_n^j\big] \end{aligned}\tag{4.16}$$

(summation convention). The symbol $\mathcal{R}_{i,j}$ denotes the pseudodifferential operator

$$\mathcal{R}_{i,j}=\partial_{x_i}\Delta^{-1}\partial_{x_j}$$

or, in terms of symbols,

$$\mathcal{R}_{i,j}[v]=\mathcal{F}^{-1}\left[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}[v](\xi)\right],$$

where $\mathcal{F}$ denotes the Fourier transform in the x-variable.

The effective viscous flux enjoys certain weak compactness properties discovered by Lions [62] which represent the key point in the global existence proof for barotropic flows. Following [62] we can use (formally) (4.16) to obtain

$$\begin{aligned} &\big(p_n-(\lambda+2\mu)\operatorname{div}\mathbf{u}_n\big)b(\varrho_n)\\ &\quad=b(\varrho_n)\Delta^{-1}\operatorname{div}(\varrho_n\mathbf{f}_n)-\partial_t\big(b(\varrho_n)\Delta^{-1}\operatorname{div}(\varrho_n\mathbf{u}_n)\big)\\ &\qquad+b(\varrho_n)_t\Delta^{-1}\operatorname{div}(\varrho_n\mathbf{u}_n)-b(\varrho_n)\mathcal{R}_{i,j}\big[\varrho_n u_n^i u_n^j\big], \end{aligned}\tag{4.17}$$

where b is as in (3.3). One should keep in mind that ϱ_n, $\mathbf{u}_n$ here are defined on a bounded domain Ω and, consequently, a localization procedure is needed to justify this argument.

Using Proposition 4.2 we can now pass to the limit for $n\to\infty$ in (4.16) and multiply the resulting expression by $\overline{b(\varrho)}$ to deduce

$$\begin{aligned} &\big(p-(\lambda+2\mu)\operatorname{div}\mathbf{u}\big)\overline{b(\varrho)}\\ &\quad=\overline{b(\varrho)}\Delta^{-1}\operatorname{div}(\varrho\mathbf{f})-\partial_t\big(\overline{b(\varrho)}\Delta^{-1}\operatorname{div}(\varrho\mathbf{u})\big)\\ &\qquad+\overline{b(\varrho)}_t\Delta^{-1}\operatorname{div}(\varrho\mathbf{u})-\overline{b(\varrho)}\mathcal{R}_{i,j}\big[\varrho u^i u^j\big]. \end{aligned}\tag{4.18}$$

Assuming, in addition to the hypotheses already made, that ϱ_n, $\mathbf{u}_n$ are also renormalized solutions of (2.1), we can use (4.7), (4.18) together with the smoothing properties of Δ^{-1}, to pass to the limit in (4.17) for $n \to \infty$ to obtain:

$$\begin{aligned}
&\lim_{n\to\infty} \int_I \int_\Omega \big(p_n - (\lambda + 2\mu)\operatorname{div}\mathbf{u}_n\big) b(\varrho_n)\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_I \int_\Omega \big(p - (\lambda + 2\mu)\operatorname{div}\mathbf{u}\big)\overline{b(\varrho)}\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \lim_{n\to\infty} \int_I \int_\Omega b(\varrho_n)\big(u_n^i \mathcal{R}_{i,j}\big[\varrho_n u_n^j\big] - \mathcal{R}_{i,j}\big[\varrho_n u_n^i u_n^j\big]\big)\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_I \int_\Omega \overline{b(\varrho)}\big(u^i \mathcal{R}_{i,j}\big[\varrho u^j\big] - \mathcal{R}_{i,j}\big[\varrho u^i u^j\big]\big)\varphi \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{4.19}$$

for any test function $\varphi \in \mathcal{D}(I \times \Omega)$.

It is a remarkable result of Lions [62] that the right-hand side of (4.19) is in fact zero. To see this we offer two rather different techniques in hope to illuminate a bit the connection of such a result with the theory of compensated compactness. Following the proof in [62] one can make use of the regularity properties of the commutator

$$u^i \mathcal{R}_{i,j}\big[\varrho u^j\big] - \mathcal{R}_{i,j}\big[\varrho u^i u^j\big]$$

discovered by Coifman and Meyer [13]. Specifically, this quantity belongs to the Sobolev space $W^{1,q}$ provided $u^i \in W^{1,2}$ and $\varrho u^j \in L^r$ with $r > 2$ in which case $1/q = 1/p + 1/2$. Of course, this hypothesis requires $\varrho \in L^p$ with $p > 3$ for $N \geqslant 3$ which is too strong for our purposes, but a simple interpolation argument shows one can treat the general case $\varrho \in L^p$, $p > N/2$, by the same method.

Pursuing a different path we can write

$$\begin{aligned}
&\int_I \int_\Omega b(\varrho_n)\big(u_n^j \mathcal{R}_{i,j}\big[\varrho_n u_n^i\big] - \mathcal{R}_{i,j}\big[\varrho_n u_n^i u_n^j\big]\big)\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_I \int_\Omega u_n^j \big(\mathbf{X}^j(\varrho_n) \cdot \mathbf{Y}(\varrho_n \mathbf{u}_n) - \mathbf{U}(\varrho_n \mathbf{u}_n) \cdot \mathbf{V}^j(\varrho_n)\big) \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where the vector fields $\mathbf{X}^j$, $\mathbf{Y}$, $\mathbf{U}$, $\mathbf{V}^j$ are given by formulas

$$\begin{aligned}
&X_k^j(\varrho_n) = \big(\varphi b(\varrho_n)\delta_{j,k} - \mathcal{R}_{k,l}\big[\varphi b(\varrho_n)\delta_{l,j}\big]\big), \qquad Y_k(\varrho_n \mathbf{u}_n) = \mathcal{R}_{k,i}\big[\varrho_n u_n^i\big], \\
&U_k(\varrho_n \mathbf{u}_n) = \varrho_n u_n^k - \mathcal{R}_{k,i}\big[\varrho_n u_n^i\big], \\
&V_k^j(\varrho_n) = \mathcal{R}_{k,l}\big[\varphi b(\varrho_n)\delta_{l,j}\big], \quad j = 1, \ldots, N,
\end{aligned}$$

and $\delta_{i,j}$ stands for the Kronecker symbol.

Now, it is easy to check that

$$\operatorname{div} \mathbf{X}^j = \operatorname{div} \mathbf{U} = 0 \quad \text{and} \quad \mathbf{Y} = \nabla\big(\Delta^{-1}\big[\operatorname{div}(\varrho_n \mathbf{u}_n)\big]\big),$$
$$\mathbf{V}^j = \nabla\big(\Delta^{-1}\big[\operatorname{div}(\varphi b(\varrho_n)\delta_{l,j})\big]\big),$$

i.e.,

$$\operatorname{curl} \mathbf{Y} = \operatorname{curl} \mathbf{V}^j = 0.$$

Applying the L^p–L^q version of Div-Curl Lemma of the compensated compactness theory (cf. Murat [78] or Yi [111]) together with (4.6), (4.13), we conclude

$$\mathbf{X}^j(\varrho_n) \cdot \mathbf{Y}(\varrho_n \mathbf{u}_n) \to \big(\varphi\overline{b(\varrho)}\delta_{j,k} - \mathcal{R}_{k,l}\big[\varphi\overline{b(\varrho)}\delta_{l,j}\big]\big) \cdot \mathcal{R}_{k,i}\big[\varrho u^i\big]$$
$$\text{in } L^2\big(I; W^{-1,2}(\Omega)\big),$$

and, similarly,

$$\mathbf{U}(\varrho_n \mathbf{u}_n) \cdot \mathbf{V}^j(\varrho_n) \to \big(\varrho u^k - \mathcal{R}_{k,i}\big[\varrho u^i\big]\big)\mathcal{R}_{k,l}\big[\varphi\overline{b(\varrho)}\delta_{l,j}\big] \quad \text{in } L^2\big(I; W^{-1,2}(\Omega)\big)$$

provided $p > N/2$. This yields, similarly as above, the desired conclusion, namely, the right-hand side of (4.19) equals zero.

Thus we have obtained the following important result (see [62]):

PROPOSITION 4.3. *Let the quantities ϱ_n, $\mathbf{u}_n$, and f_n satisfy the hypotheses* (4.8), (4.9) *for $p > N/2$, together with* (4.14), (4.15). *Let, moreover,*

$$\|p_n\|_{L^r(I\times\Omega)} \leqslant c \quad \text{for a certain } r > 1.$$

Then we have

$$\lim_{n\to\infty} \int_I \int_\Omega \big(p_n - (\lambda + 2\mu)\operatorname{div} \mathbf{u}_n\big) b(\varrho_n)\varphi \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_I \int_\Omega \big(p - (\lambda + 2\mu)\operatorname{div} \mathbf{u}\big)\overline{b(\varrho)}\varphi \,\mathrm{d}x\,\mathrm{d}t$$

for any b satisfying (3.3) *and any test function $\varphi \in \mathcal{D}(I \times \Omega)$.*

4.4. *Bibliographical remarks*

The theory of compensated compactness has played a crucial role in the development of the first large data existence results for systems of nonlinear conservation laws (see Dafermos [14], DiPerna [21], Tartar [102]). A good survey on weak convergence methods can be found in the monograph by Evans [26]. One of the well-known results is the so-called Div-Curl Lemma refered to above (cf. Murat [78]):

LEMMA 4.1. *Let* $\mathbf{U}_n$, $\mathbf{V}_n$ *be two sequences of vector functions defined on some open set* $Q \subset R^N$ *such that*

$$\mathbf{U}_n \to \mathbf{U} \quad \textit{weakly in } L^p(Q), \qquad \mathbf{V}_n \to \mathbf{V} \quad \textit{weakly in } L^q(Q);$$

and

$$\operatorname{div} U_n \textit{ precompact in } W^{-1,p}(Q), \qquad \operatorname{curl} V_n \textit{ precompact in } W^{-1,q}(Q),$$

where

$$1 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} \leqslant 1.$$

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \quad \textit{in } \mathcal{D}'(Q).$$

Note that the situation in Proposition 4.3 is particularly simple as $\operatorname{div} U_n = \operatorname{curl} V_n = 0$ and the proof of Lemma 4.1 is elementary.

There is yet another way to show Proposition 4.3 presented in [27, Lemma 5].

The defect measures similar to osc were introduced by DiPerna and Majda [24] in their study of the Euler equations.

5. Mathematical theory of barotropic flows

We review the recent development of the mathematical theory of barotropic flows, specifically, we shall discuss some large data existence and related results originated by the pioneering work of Lions [62]. Accordingly, the crucial hypothesis we cannot dispense with is that the pressure p and the density ϱ are functionally dependent and the relation between them is given by formula

$$p = p(\varrho) \quad \text{with } p:[0,\infty) \to [0,\infty) \text{ – a nondecreasing and continuous function.} \tag{5.1}$$

As a matter of fact, most of the results will be stated for the simpler isentropic pressure-density relation

$$p(\varrho) = a\varrho^{\gamma}, \quad a > 0,\ \gamma \geqslant 1, \tag{5.2}$$

and possible generalizations discussed afterwards.

The temperature θ being eliminated from the pressure constitutive law, the system (2.1)–(2.3) reduces to

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{5.3}$$

$$\frac{\partial \varrho \mathbf{u}}{\partial t} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla(\operatorname{div} \mathbf{u}) + \varrho \mathbf{f}. \tag{5.4}$$

The spatial variable x will belong to a regular *bounded* domain $\Omega \subset R^N$, $N = 2, 3$, and the velocity $\mathbf{u}$ will satisfy the no-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{5.5}$$

Taking (formally) the scalar product of (5.4) with $\mathbf{u}$ and integrating by parts we obtain the energy inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(t)\,\mathrm{d}x + \int_\Omega \mu|\nabla \mathbf{u}|^2 + (\lambda+\mu)|\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x \leqslant \int_\Omega \varrho \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x, \tag{5.6}$$

where the specific energy E satisfies (1.8). If p is given by (5.2), we have

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho\log(\varrho) \quad \text{for } \gamma = 1, \qquad E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma \quad \text{if } \gamma > 1.$$

As already agreed on in Section 1, the fluids under consideration are viscous, i.e.,

$$\mu > 0 \quad \text{and} \quad \lambda + \mu \geqslant 0.$$

Note that the restrictions imposed on λ allow for all physically relevant situations.

In what follows, we consider the *finite energy weak solutions* of the problem (5.3)–(5.5) on the set $I \times \Omega$, more specifically, ϱ, $\mathbf{u}$ will meet the following set of conditions:

- the density ϱ and the velocity $\mathbf{u}$ satisfy

$$\varrho \geqslant 0, \qquad \varrho \in L^\infty\big(I; L^\gamma(\Omega)\big), \qquad \mathbf{u} \in L^2\big(I; \big[W_0^{1,2}(\Omega)\big]^N\big);$$

- the specific energy E belongs to $L^1_{\mathrm{loc}}(I; L^1(\Omega))$ and the energy inequality (5.6) holds in $\mathcal{D}'(I)$, i.e.,

$$\int_I \partial_t\psi\left(\int_\Omega E\,\mathrm{d}x\right)\mathrm{d}t - \int_I \psi\left(\int_\Omega \mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x\right)\mathrm{d}t \geqslant \int_I \psi \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t$$

  holds for any function $\psi \in \mathcal{D}(I)$, $\psi \geqslant 0$;

- the functions ϱ, $\mathbf{u}$ extended to be zero outside Ω solve the continuity equation (5.3) in $\mathcal{D}'(I \times R^N)$ (cf. (2.4)); moreover, (5.3) is satisfied in the sense of renormalized solutions, i.e., (3.2) holds for any b as in (3.3);
- the equations of motion (5.4) are satisfied in $\mathcal{D}'(I \times \Omega)$ (cf. (2.5)).

As the reader will have noticed in Section 4, the value of the adiabatic constant γ will play an important role in the analysis. In most cases, we shall assume

$$\gamma > N/2,$$

where $N = 2, 3$ are the physically relevant situations.

The external force density $\mathbf{f}$ is assumed to be a bounded and measurable function such that

$$\operatorname{ess}\sup_{t\in I,\ x\in\Omega} \left|\mathbf{f}(t,x)\right| \leqslant F.$$

In what follows, we shall give an outline of the large data existence results in the class of finite energy weak solutions. We shall also discuss the long-time behaviour and related asymptotic problems. To this end, we pursue the classical scheme for solving nonlinear problems:

- First of all, we find *a priori estimates*, i.e., the bounds imposed formally on any classical solution and depending only on the data (cf. Sections 5.1, 5.2).
- Given a family of solutions satisfying the bounds induced by a priori estimates, we examine the question of *compactness*, i.e., whether or not any accummulation point of this family in suitable topologies is again a solution of the original problem (see Sections 5.3, 5.4).
- Finally, one has to find a suitable *approximation scheme* solvable, say, by a classical fixed-point technique, and compatible with both the estimates and compactness properties mentioned above (Section 5.5).

To conclude this introduction, let us note that any finite energy weak solution satisfies

$$\begin{aligned} &\varrho \in C\big(I; L^{\gamma}_{\text{weak}}(\Omega)\big) \cap C\big(I; L^{\alpha}(\Omega)\big), \quad 1 \leqslant \alpha < \gamma, \\ &\varrho\mathbf{u} \in C\big(I; L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega)\big) \end{aligned} \tag{5.7}$$

provided $\gamma > N/2$ (cf. Proposition 3.1). In particular, the density and the momenta are well defined at *any* specific time $t \in I$. Moreover, the *total mass*

$$m = \int_{\Omega} \varrho \,\mathrm{d}x \quad \text{is independent of } t \in I; \tag{5.8}$$

and the *total energy* $\mathcal{E}$ defined for any $t \in I$ by formula

$$\mathcal{E}(t) = \mathcal{E}\big[\varrho, (\varrho\mathbf{u})\big](t) = \int_{\varrho(t)>0} \frac{1}{2}\frac{|(\varrho\mathbf{u})|^2}{\varrho}(t) + \frac{a}{\gamma-1}\varrho^{\gamma}(t)\,\mathrm{d}x \tag{5.9}$$

is a lower semi-continuous function of $t \in I$ (see [27, Corollary 2]).

5.1. *Energy estimates*

Besides the total mass m, the total energy $\mathcal{E}$ is another quantity which can be shown bounded in terms of the data at least on compact time intervals.

PROPOSITION 5.1. *Let $\Omega \subset R^N$ be a bounded Lipschitz domain. Let ϱ, $\mathbf{u}$ be a finite energy weak solution of* (5.3)–(5.5) *where the pressure satisfies the isentropic constitutive law* (5.2) *with $\gamma > N/2$.*

Then

$$\|\varrho(t)\|^{\gamma}_{L^{\gamma}(\Omega)} + \int_{\Omega} \varrho(t)\left|\mathbf{u}(t)\right|^2 \mathrm{d}x + \int_{\inf\{I\}}^{t} \int_{\Omega} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant c\big(\mathcal{E}_0, m, F, t - \inf\{I\}\big), \tag{5.10}$$

where the quantity c is bounded for bounded values of arguments and

$$\mathcal{E}_0 = \limsup_{t \to \inf\{I\}+} \mathcal{E}\big[\varrho, (\varrho\mathbf{u})\big](t).$$

The bound (5.10), which can be easily obtained combining the energy inequality (5.6) and the Gronwall lemma, can be viewed as an a priori estimate though it holds for any finite energy weak solution of the problem. It is not difficult to see that similar results can be derived provided p is given by a general constitutive relation (5.1) and satisfies suitable growth conditions for large values of the density. On the other hand, as already mentioned in Section 2.5, *uniform* a priori estimates of ϱ seems to be out of reach of the standard techniques and represent a major open problem of the present theory.

5.2. *Pressure estimates for isentropic flows*

By virtue of (5.10), the isentropic pressure $p(\varrho)$ belongs automatically to the set $L^1(I \times \Omega)$ at least for bounded time intervals I. On the other hand, the weak compactness results like Proposition 4.3 require p in a weakly complete (reflexive) space $L^r(I \times \Omega)$ with $r > 1$. Such a bound is indeed available as the following result shows:

PROPOSITION 5.2. *Assume $\Omega \subset R^N$, $N \geqslant 2$, is a bounded Lipschitz domain. Let ϱ, $\mathbf{u}$ be a finite energy weak solution to the problem* (5.3)–(5.5) *on $I \times \Omega$ where the isentropic pressure p is given by* (5.2) *with $\gamma > N/2$. Let*

$$0 < \eta < \min\left\{\frac{1}{4}, \frac{1}{\gamma}\left(\frac{2\gamma}{N} - 1\right)\right\}$$

be given. Denote by $m = \int_{\Omega} \varrho \,\mathrm{d}x$ the (*conserved*) *total mass and let $F = \operatorname{ess\,sup}_{I \times \Omega} |\mathbf{f}|$.*

Then for any bounded time interval $J \subset I$, we have

$$\int_J \int_\Omega \varrho^{\gamma+\eta} \, dx \, dt \leqslant c\big(m, F, \eta, |J|\big)\Big(1 + \sup_{t \in J} \mathcal{E}(t)\Big)^{\frac{\gamma+1}{\gamma}}. \tag{5.11}$$

A local version of the above estimates was obtained by Lions [62]. In fact, the bounds on η in Proposition 5.2 are not optimal. Similarly as in the local case (see [62]), one could verify the best values for η:

$$0 < \eta \leqslant \frac{2}{N}\gamma - 1.$$

However, for further purposes, it is convenient to have the integrals containing ϱ^η bounded in terms of the total mass m equivalent to the L^1-norm rather than the total energy $\mathcal{E}$ proportional to the L^γ-norm of ϱ.

The validity of (5.11) up to the boundary of Ω was proved in [39] by means of a multiplier technique. More specifically, the main idea is to take the quantities

$$\phi_i(t, x) = \psi(t)\mathcal{B}_i\big[\vartheta_\varepsilon * T_k(\varrho^\eta)\big], \quad i = 1, \ldots, N,$$

as test functions for (2.5). Here

$$\psi \in \mathcal{D}(I),\ 0 \leqslant \psi \leqslant 1, \qquad \int_I |\partial_t \psi| \, dt \leqslant 2,$$

$\vartheta_\varepsilon(x)$ is a regularizing sequence as in Lemma 3.1, T_k are the cut-off operators introduced in (4.5), and, most importantly, the symbol $\mathcal{B}$ stands for an inverse of the div operator, i.e., $\mathbf{v} = \mathcal{B}[g]$ solves the equation

$$\operatorname{div} \mathbf{v} = g - \frac{1}{|\Omega|}\int_\Omega g \, dx, \qquad \mathbf{v}|_{\partial\Omega} = 0. \tag{5.12}$$

Equation (5.12) has been studied by many authors. Here, we have adopted the approach which is essentially due to Bogovskii [8]. It can be shown that the problem (5.12) admits a solutions operator $\mathcal{B}: g \mapsto \mathbf{v}$ enjoying the following properties:

- $\mathcal{B} = [\mathcal{B}_1, \ldots, \mathcal{B}_N]$ is a bounded linear operator from $L^p(\Omega)$ into $[W_0^{1,p}(\Omega)]^N$, specifically,

$$\big\|\mathcal{B}[g]\big\|_{W_0^{1,p}(\Omega)} \leqslant c(p)\|g\|_{L^p(\Omega)} \quad \text{for any } 1 < p < \infty.$$

- The function $\mathbf{v} = \mathcal{B}[g]$ solves the problem (5.12).
- If, moreover, $g \in L^p(\Omega)$ can be written in the form $g = \operatorname{div} \mathbf{h}$ where $\mathbf{h} \in [L^r(\Omega)]^N$, $\mathbf{h} \cdot \mathbf{n} = 0$ on $\partial\Omega$, then

$$\big\|\mathcal{B}[g]\big\|_{L^r(\Omega)} \leqslant c(p, r)\|\mathbf{h}\|_{L^r(\Omega)}.$$

The proof of the existence of the operator $\mathcal{B}$ as well as the above properties can be found in Galdi [42] or Borchers and Sohr [9].

An alternative approach to show (5.11) based on properties of the Stokes operator was proposed by Lions [63].

The estimates (5.8), (5.10), (5.11) are the only available for the problem (5.3)–(5.5) unless some smallness assumptions are imposed on the data, i.e., on $\mathcal{E}_0$, m, and $\mathbf{f}$. Thus in accordance with Proposition 4.2, the main stumbling block to show compactness of solutions is the pressure term. Indeed the above estimates guarantee only weak compactness of the density ϱ while for $p = p(\varrho)$ to be compact we need strong compactness of ϱ in L^p. As we shall see later, the way out this vicious circle is provided by Proposition 4.3, namely, by the compactness properties of the effective viscous flux.

5.3. *Density oscillations for barotropic flows*

We show how Proposition 4.3 can be used to describe the amplitude of density oscillations for barotropic flows discussed in Section 4.1.

In addition to (5.1), we suppose

$$p(0) = 0, \qquad p(\varrho) \geqslant 0 \quad \text{for } \varrho \geqslant 0, \; p \text{ convex on } [0, \infty).$$

It is easy to verify that

$$p(y) - p(z) \geqslant p(y - z) \quad \text{for all } 0 \leqslant z \leqslant y$$

which yields immediately

$$\begin{aligned} &\big(p(y) - p(z)\big)\big(T_k(y) - T_k(z)\big) \\ &\quad \geqslant p\big(\big|T_k(y) - T_k(z)\big|\big)\big|T_k(y) - T_k(z)\big| \quad \text{for all } y, z \geqslant 0. \end{aligned} \tag{5.13}$$

Under the hypotheses (and notation) of Proposition 4.3, we have $p = \overline{p(\varrho)}$ and we can write

$$\begin{aligned} &\lim_{n\to\infty} \int_Q p(\varrho_n) T_k(\varrho_n) - \overline{p(\varrho)}\,\overline{T_k(\varrho)} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \lim_{n\to\infty} \int_Q \big(p(\varrho_n) - p(\varrho)\big)\big(T_k(\varrho_n) - T_k(\varrho)\big) \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_Q \big(\overline{p(\varrho)} - p(\varrho)\big)\big(T_k(\varrho) - \overline{T_k(\varrho)}\big) \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{5.14}$$

for any bounded $Q \subset I \times \Omega$.

As p is convex and T_k concave on $[0,\infty)$, the second integral on the right-hand side of (5.14) is non-negative and, making use of (5.13) we infer

$$\lim_{n\to\infty}\int_Q p(\varrho_n)T_k(\varrho_n) - \overline{p(\varrho)}\,\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t$$

$$\geqslant \limsup_{n\to\infty}\int_Q p\big(\big|T_k(\varrho_n)-T_k(\varrho)\big|\big)\big|T_k(\varrho_n)-T_k(\varrho)\big|\,\mathrm{d}x\,\mathrm{d}t. \tag{5.15}$$

On the other hand, we have

$$\lim_{n\to\infty}\int_Q \operatorname{div}\mathbf{u}_n T_k(\varrho_n) - \operatorname{div}\mathbf{u}\overline{T_k(\varrho)}\,\mathrm{d}x\,\mathrm{d}t$$

$$= \lim_{n\to\infty}\int_Q \operatorname{div}\mathbf{u}_n\big(T_k(\varrho_n)-\overline{T_k(\varrho)}\big)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant \sup_{n\geqslant 1}\|\operatorname{div}\mathbf{u}_n\|_{L^2(Q)}\limsup_{n\to\infty}\big\|T_k(\varrho_n)-\overline{T_k(\varrho)}\big\|_{L^2(Q)}. \tag{5.16}$$

Thus if the pressure is superlinear at infinity, the relations (5.15), (5.16) together with Proposition 4.3 enable to estimate the amplitude of oscillations $\operatorname{osc}_p[\varrho_n-\varrho](Q)$ introduced in Section 4.1. In particular, the following result holds (see [29, Proposition 6.1]).

PROPOSITION 5.3. *Let $\Omega\subset R^N$, $N\geqslant 2$ be a bounded Lipschitz domain and $I\subset R$ a bounded interval. Let the pressure p be given by the formula* (5.1), *$p(0)=0$, p convex, and*

$$p(\varrho)\geqslant a\varrho^\gamma,\quad a>0,\ \gamma>N/2.$$

Let ϱ_n, $\mathbf{u}_n$ be a sequence of finite energy weak solutions of the problem (5.3)–(5.5) *with $\mathbf{f}=\mathbf{f}_n$ and such that*

$$m_n=\int_\Omega \varrho_n\leqslant m,$$

$$\limsup_{t\to\inf\{I\}+}\mathcal{E}\big[\varrho_n,(\varrho_n\mathbf{u}_n)\big](t)\leqslant\mathcal{E}_0,$$

and

$$\operatorname*{ess\,sup}_{I\times\Omega}|\mathbf{f}_n|\leqslant F$$

independently of n.

Then

$$\operatorname{osc}_{\gamma+1}[\varrho_n-\varrho](Q)\leqslant c(Q)\Big(\sup_{n\geqslant 1}\|\operatorname{div}\mathbf{u}_n\|_{L^2(Q)}\Big)^{1/\gamma}$$

for any weak limit ϱ of the sequence ϱ_n and any bounded $Q \subset I \times \Omega$.

As a straightforward consequence of Propositions 4.1, 5.3 we get the following:

COROLLARY 5.1. *Under the hypotheses of Proposition* 5.3, *let*

$$\varrho_n \to \varrho \quad \text{weakly star in } L^\infty\big(I; L^\gamma(\Omega)\big),$$
$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } L^2\big(I; W_0^{1,2}(\Omega)\big).$$

Then ϱ, $\mathbf{u}$ solve (5.3) *in the sense of renormalized solutions, i.e., Equation* (3.2) *holds for any b satisfying* (3.3).

5.4. *Propagation of oscillations*

For simplicity, we suppose the pressure p is given by the isentropic constitutive relation (5.2) with $\gamma > N/2$. Similarly as in Proposition 5.3, let ϱ_n, $\mathbf{u}_n$ be a sequence of finite energy weak solutions of (5.3)–(5.5) on some bounded time interval I such that

$$\limsup_{t \to \inf\{I\}+} \mathcal{E}\big[\varrho_n, (\varrho_n \mathbf{u}_n)\big] \leqslant \mathcal{E}_0, \qquad \|\mathbf{f}_n\|_{L^\infty(I\times\Omega)} \leqslant F$$

uniformly in n.

The issue we want to address now is the time propagation of oscillations in the density component. To begin with, it seems worth-observing that any reasonable solution operator we could associate with the finite energy weak solutions cannot be compact with respect to ϱ. This is due to the hyperbolic character of the continuity equation (5.3). In accordance with the observations made by Lions [60], the oscillations should propagate in time. Serre [92] studied this phenomenon and showed the amplitude of the Young measures associated to the sequence $\varrho_n(t)$ is a non-increasing function of time. His proof is complete in the dimension $N = 1$ and formal for $N \geqslant 2$ taking the conclusion of Proposition 4.3 for granted. Having proved Proposition 4.3 Lions [62] completed the proof for $N \geqslant 2$. The fact that oscillations cannot be created in ϱ_n unless they were present initially plays the crucial role in the existence theory developed in [62].

Here we go a step further by showing that the amplitude of possible oscillations decays with time at uniform rate depending solely on the value of the initial energy $\mathcal{E}_0$ (see [37]). In particular, the time images of bounded energy initial data are asymptotically compact with respect to the density component. This is precisely what is needed to develop a meaningful dynamical systems theory associated to the problem.

In accordance with our hypotheses, we can show

$$\varrho_n \to \varrho \quad \text{in } C\big(I; L^\gamma_{\text{weak}}(\Omega)\big),$$
$$T_k(\varrho_n) \to \overline{T_k(\varrho)} \quad \text{in } C\big(I; L^\alpha_{\text{weak}}(\Omega)\big) \text{ for any } \alpha \geqslant 1,\ k \geqslant 1,$$
$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } L^2\big(I; W_0^{1,2}(\Omega)\big).$$

To measure the amplitude of oscillations of the sequence ϱ_n, we introduce a *defect measure* dft,

$$\mathrm{dft}[\varrho_n - \varrho](t) = \int_\Omega \nu(t,x)\,\mathrm{d}x, \quad \text{where } \nu = \overline{\varrho\log(\varrho)} - \varrho\log(\varrho). \tag{5.17}$$

By virtue of Corollary 5.1, both ϱ_n and ϱ are renormalized solutions of (5.3) and we have

$$\begin{aligned}
&\varrho_n\log(\varrho_n) \to \overline{\varrho\log(\varrho)} \quad \text{in } C\big(J; L^\alpha_{\mathrm{weak}}(\Omega)\big),\ 1 \leqslant \alpha < \gamma,\\
&\varrho\log(\varrho) \in C\big(J; L^\alpha_{\mathrm{weak}}(\Omega)\big), \quad 1 \leqslant \alpha < \gamma.
\end{aligned}$$

Consequently, $\mathrm{dft}[\varrho_n - \varrho]$ is a continuous function of $t \in I$.

Mainly for technical reasons, we are not able to deal directly with the function dft. We consider instead a family of approximate functions:

$$L_k(z) = \begin{cases} z\log(z) & \text{for } 0 \leqslant z \leqslant k,\\ z\log(k) + z\int_z^k T_k(s)/s^2\,\mathrm{d}s & \text{for } z \geqslant k. \end{cases}$$

It is easy to observe that $L_k(z) = \beta_k z + b_k(z)$ where b_k satisfy (3.3) and

$$L_k'(z)z - L_k(z) = T_k(z).$$

Since both ϱ_n, ϱ are renormalized solutions of (5.3) on $I \times R^3$, we deduce

$$\begin{aligned}
&\int_\Omega \big(L_k(\varrho_n) - L_k(\varrho)\big)(t_2)\,\mathrm{d}x - \int_\Omega \big(L_k(\varrho_n) - L_k(\varrho)\big)(t_1)\,\mathrm{d}x\\
&\quad = \int_{t_1}^{t_2}\int_\Omega T_k(\varrho)\,\mathrm{div}\,\mathbf{u} - \overline{T_k(\varrho)}\,\mathrm{div}\,\mathbf{u}_n\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \int_{t_1}^{t_2}\int_\Omega \big(\overline{T_k(\varrho)} - T_k(\varrho_n)\big)\,\mathrm{div}\,\mathbf{u}_n\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for any $t_1, t_2 \in I$.

Letting $n \to \infty$ and using Proposition 4.3 together with (5.15), we obtain

$$\begin{aligned}
&\int_\Omega \big(\overline{L_k(\varrho)} - L_k(\varrho)\big)(t_2)\,\mathrm{d}x - \int_\Omega \big(\overline{L_k(\varrho)} - L_k(\varrho)\big)(t_1)\,\mathrm{d}x\\
&\qquad + \frac{a}{\lambda + 2\mu}\limsup_{n\to\infty}\int_{t_1}^{t_2}\int_\Omega \big|T_k(\varrho_n) - T_k(\varrho)\big|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad \leqslant \int_{t_1}^{t_2}\int_\Omega \big(T_k(\varrho) - \overline{T_k(\varrho)}\big)\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \quad \text{for any } t_1 \leqslant t_2,\ t_1, t_2 \in I.
\end{aligned} \tag{5.18}$$

Our aim now is to pass to the limit for $k \to \infty$ in (5.18). Clearly,

$$\int_\Omega \big(\overline{L_k(\varrho)} - L_k(\varrho)\big)(t)\,\mathrm{d}x\,\mathrm{d}t \to \mathrm{dft}[\varrho_n - \varrho](t) \quad \text{for } k \to \infty$$

while

$$\begin{aligned}&\big\|T_k(\varrho) - \overline{T_k(\varrho)}\big\|_{L^2(I\times\Omega)}\\ &\quad\leqslant \big\|T_k(\varrho) - \overline{T_k(\varrho)}\big\|^{\beta}_{L^1(I\times\Omega)}\big(\mathrm{osc}_{\gamma+1}[\varrho_n - \varrho](I\times\Omega)\big)^{1-\beta}, \quad \beta = \frac{\gamma-1}{2\gamma}.\end{aligned}$$

By virtue of Proposition 5.3, the right-hand side of the above inequality tends to zero for $k \to \infty$ and so does the right-hand side of (5.18).

Finally, we have

$$\begin{aligned}&\limsup_{n\to\infty}\int_{t_1}^{t_2}\int_\Omega \big|T_k(\varrho_n) - T_k(\varrho)\big|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\geqslant |\Omega|^{\frac{\alpha-\gamma+1}{\alpha}}\limsup_{n\to\infty}\int_{t_1}^{t_2}\big\|T_k(\varrho_n) - T_k(\varrho)\big\|^{\gamma+1}_{L^\alpha(\Omega)}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and (5.18) yields:

$$\begin{aligned}&\mathrm{dft}[\varrho_n - \varrho](t_2) - \mathrm{dft}[\varrho_n - \varrho](t_1)\\ &\quad+ \left(\frac{a|\Omega|^{\frac{\alpha-\gamma+1}{\alpha}}}{\lambda + 2\mu}\right)\limsup_{n\to\infty}\int_{t_1}^{t_2}\|\varrho_n - \varrho\|^{\gamma+1}_{L^\alpha(\Omega)}\,\mathrm{d}x\,\mathrm{d}t \leqslant 0 \qquad (5.19)\end{aligned}$$

for any $t_1 \leqslant t_2$, $t_1, t_2 \in I$ and $1 \leqslant \alpha < \gamma$.

To conclude, we shall need the following auxiliary result (cf. [28, Lemma 2.1]).

LEMMA 5.1. *Given $\alpha \in (1, \gamma)$ there exists $c = c(\alpha)$ such that*

$$z\log(z) - y\log(y) \leqslant \big(1 + \log^+(y)\big)(z - y) + c(\alpha)\big(|z-y|^{1/2} + |z-y|^\alpha\big)$$

for any $y, z \geqslant 0$.

In accordance with Lemma 5.1, we can write

$$\begin{aligned}&\int_\Omega \varrho_n\log(\varrho_n)\,\mathrm{d}x - \int_\Omega\big(1 + \log^+(\varrho)\big)(\varrho_n - \varrho)\,\mathrm{d}x\\ &\quad\leqslant c(\alpha)\big(|\Omega|^{\frac{2\alpha-1}{2\alpha}}\|\varrho_n - \varrho\|^{2\alpha}_{L^\alpha(\Omega)} + \|\varrho_n - \varrho\|^{\alpha}_{L^\alpha(\Omega)}\big)\end{aligned}$$

which together with (5.19) yields

$$\mathrm{dft}[\varrho_n - \varrho](t_2) - \mathrm{dft}[\varrho_n - \varrho](t_1) + \int_{t_1}^{t_2}\Phi\big(\mathrm{dft}[\varrho_n - \varrho](t)\big)\,\mathrm{d}t \leqslant 0,$$

where the nonlinear function Φ depends only on the structural properties of the logarithm and can be chosen independently of the data to satisfy

$$\Phi : R \mapsto R \quad \text{is continuous and strictly increasing, } \Phi(0) = 0. \tag{5.20}$$

Summing up the above considerations we have arrived at the following conclusion:

PROPOSITION 5.4. *Let $\Omega \subset R^N$, $N \geqslant 2$ be a bounded Lipschitz domain and $I \subset R$ a bounded interval. Let ϱ_n, $\mathbf{u}_n$ be a sequence of finite energy weak solutions of the problem* (5.3)–(5.5) *on $I \times \Omega$, where pressure p is given by the isentropic constitutive relation*

$$p = a\varrho^\gamma, \quad a > 0,\ \gamma > \frac{N}{2},$$

and $\mathbf{f} = \mathbf{f}_n$. Let

$$\limsup_{t \to \inf\{I\}+} \mathcal{E}\big[\varrho_n, (\varrho_n \mathbf{u}_n)\big](t) \leqslant \mathcal{E}_0, \qquad \|\mathbf{f}_n\|_{L^\infty(I\times\Omega)} \leqslant F$$

independently of n. Let ϱ be a weak limit of the sequence ϱ_n.

Then

$$\operatorname{dft}[\varrho_n - \varrho](t_2) \leqslant \chi(t_2 - t_1) \quad \textit{for any } t_1, t_2 \in I,\ t_1 \leqslant t_2,$$

where χ is the unique solution of the initial-value problem

$$\chi'(t) + \Phi\big(\chi(t)\big) = 0, \quad \chi(0) = \operatorname{dft}[\varrho_n - \varrho](t_1)$$

and Φ is a fixed function satisfying (5.20).

It can be shown that Φ has a polynomial growth for values close to zero and, consequently, the quantity $\operatorname{dft}[\varrho_n - \varrho](t)$ behaves like $t^{-\beta}$ for a certain $\beta > 0$ when $t \to \infty$.

5.5. *Approximate solutions*

The *a priori* estimates derived in Sections 5.1, 5.2 together with the compactness results in Propositions 4.2, 5.4 form a suitable platform for a larga data existence theory for the problem (5.3)–(5.5). The final task, as usual, is to find a suitable *approximation scheme* compatible with both *a priori* estimates and the compactness results claimed above. Needless to say there are many ways to do it. Here we pursue the approach of [32] and consider the approximate problem:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{u}) = \varepsilon \Delta \varrho, \tag{5.21}$$

$$\frac{\partial \varrho \mathbf{u}}{\partial t} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) + \delta \nabla \varrho^\beta + \varepsilon \nabla \mathbf{u} \cdot \nabla \varrho$$
$$= \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \varrho \mathbf{f} \tag{5.22}$$

complemented by the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \nabla \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{5.23}$$

The parameters $\varepsilon > 0$, $\delta > 0$ are "small" and $\beta > 0$ "large". The system (5.21)–(5.23) can be solved by means of the standard Faedo–Galerkin method to obtain approximate solutions $\varrho_{\varepsilon,\delta}$, $\mathbf{u}_{\varepsilon,\delta}$ (cf. [32, Proposition 2.1]). Then one can pass to the limit, first for $\varepsilon \to 0$ and then for $\delta \to 0$, to obtain a finite energy weak solution of the problem (5.3)–(5.5) (see [32]).

The reason for introducing two parameters ε and δ is that the energy estimates presented in Section 5.1 and the pressure estimates in Section 5.2 are compatible only if $\beta > N$.

An alternative approach is the approximation scheme introduced by Lions [62] or the method of time-discretization based on solving a family of stationary problems (see also Lions [62]).

6. Barotropic flows: large data existence results

The mathematical theory presented in Section 5 can be used to obtain rigorous existence results for barotropic flows with essentially no restriction on the size of the data. We start with a very particular case posed in two space dimension where one can show even existence of strong (classical) solutions.

6.1. *Global existence of classical solutions*

The result we are going to present is due to Vaigant and Kazhikhov [107]. Consider the system

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{6.1}$$

$$\frac{\partial (\varrho u)}{\partial t} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + a \nabla \varrho^\gamma = \mu \Delta \mathbf{u} + \nabla\big((\lambda(\varrho) + \mu) \operatorname{div} \mathbf{u}\big), \tag{6.2}$$

where $(t, x) \in (0, T) \times R^2$. The functions ϱ, $\mathbf{u}$ are for simplicity considered spatially periodic, i.e.,

$$\varrho(t, x + \boldsymbol{\omega}) = \varrho(t, x), \qquad \mathbf{u}(t, x + \boldsymbol{\omega}) = \mathbf{u}(t, x). \tag{6.3}$$

The problem is complemented by the initial conditions

$$\varrho(0, x) = \varrho_0(x) \geqslant \underline{\varrho} > 0, \qquad \mathbf{u}(0, x) = \mathbf{u}_0(x). \tag{6.4}$$

Under the hypotheses

$$\mu > 0, \qquad a > 0, \qquad \gamma \geqslant 0, \quad \text{and} \quad \lambda(\varrho) = b\varrho^{\beta}, \quad b > 0,\ \beta \geqslant 3, \tag{6.5}$$

Vaigant and Kazhikhov [107] proved the following result.

THEOREM 6.1. *In addition to the hypotheses* (6.5)*, let*

$$\varrho_0 \in L^{\infty}_{\text{per}}\big(R^2\big), \qquad \mathbf{u}_0 \in W^{1,2}_{\text{per}}\big(R^2\big).$$

Then the initial-value problem (6.1)–(6.4) *possesses a global* ($T = \infty$) *weak solution. The continuity equation* (6.1) *holds in* $\mathcal{D}'((0,T) \times R^2)$ *and the equations of motion* (6.2) *are satisfied a.a. on* $(0,T) \times R^2$.

If, moreover,

$$\varrho_0 \in W^{1,q}_{\text{per}}\big(R^2\big), \qquad \mathbf{u}_0 \in W^{2,q}_{\text{per}}\big(R^2\big) \quad \textit{for some } q > 2,$$

then there is a unique strong solution satisfying the equations a.e. on $(0,T) \times R^2$.

Finally, if

$$\varrho_0 \in C^{1+\alpha}_{\text{per}}\big(R^2\big), \qquad \mathbf{u}_0 \in C^{2+\alpha}_{\text{per}}\big(R^2\big) \quad \textit{for some } \alpha > 0,$$

then the strong solution is classical (*smooth*).

Theorem 6.1 is a remarkable result since it solves both the problem of existence and uniqueness as well as regularity of solutions. The obvious restrictions of applicability are due to the rather unnatural hypotheses (6.5), i.e., the viscosity coefficient μ must be constant while λ depends on ϱ in a very specific way.

The proof of Theorem 6.1 is based on very strong *a priori* estimates – much better than presented in Sections 5.1, 5.2. These estimates are available thanks to the particular form of the constitutive relations and the fact the problem is posed in two space dimensions.

6.2. *Global existence of weak solutions*

We consider the problem (5.3)–(5.5) posed on a bounded regular domain $\Omega \subset R^N$, $N = 2, 3$. We prescribe the initial conditions

$$\varrho(0) = \varrho_0 \geqslant 0, \qquad (\varrho\mathbf{u})(0) = \mathbf{q}_0 \tag{6.6}$$

satisfying a compatibility condition

$$\mathbf{q}_0(x) = 0 \quad \text{whenever } \varrho_0(x) = 0 \tag{6.7}$$

and such that

$$\varrho_0 \in L^\gamma(\Omega), \qquad \frac{|\mathbf{q}_0|^2}{\varrho_0} \in L^1(\Omega). \tag{6.8}$$

The assumption (6.8) is nothing else but the requirement the initial data to be of finite energy.

The following theorem asserts the existence of the finite energy weak solutions to the problem (5.3)–(5.5), (6.6) introduced in Section 5.

THEOREM 6.2. *Let $\Omega \subset R^N$, $N = 2, 3$, be a bounded regular domain and $T > 0$ given. Consider the system* (5.3), (5.4) *complemented by* (5.5), (6.6)*, where p is given by the isentropic constitutive law* (5.2) *with*

$$\gamma > \frac{N}{2},$$

$\mathbf{f}$ *is a bounded measurable function on* $(0, T) \times \Omega$*, and the initial data* ϱ_0, $\mathbf{q}_0$ *satisfy* (6.7), (6.8).

Then the problem (5.3)–(5.5) *posseses a finite energy weak solution* ϱ, $\mathbf{u}$ *on* $(0, T) \times \Omega$ *satisfying the initial conditions* (6.6).

As already remarked in (5.7), both the density ϱ and the momenta $(\varrho\mathbf{u})$ are continuous functions of t with respect to the L^p-weak topology, and, consequently, the initial conditions (6.6) make sense.

The existence result stated in Theorem 6.2 was first proved by Lions [62] for $\gamma \geqslant 3/2$ if $N = 2$ and $\gamma \geqslant 9/5$ for $N = 3$. The proof needs some modifications presented in [39] and [63] to accommodate the Dirichlet boundary conditions. The present version including the full range of $\gamma > N/2$ was shown in [32, Theorem 1.1].

Given the weak compactness results, namely Propositions 4.1, 4.2, for solutions of (5.3), (5.4) respectively, the main ingredient of the proof of Theorem 6.2 is the strong compactness of the density stated in Proposition 5.4, the proof of which requires, among other things, convexity of the pressure. It is easy to see, however, that the same result can be obtained for a general barotropic pressure p as in (5.1) that can be written in the form

$$p(\varrho) = a\varrho^\gamma + p_0(\varrho), \quad a > 0,\ \gamma > N/2,$$

where p_0 is a globally Lipschitz function. Note that the proof for the range $\gamma \geqslant 9/5$, $N = 3$, $\gamma \geqslant 3/2$, $N = 2$ can be modified to include a general constitutive law (5.1) where $p(\varrho) \geqslant a\varrho^\gamma$ for all ϱ large enough (cf. Lions [62]).

It seems interesting to note that the physically relevant isothermal case where $\gamma = 1$ seems to be completely open even if $N = 2$. The only large data existence result is that of Hoff [46] where the initial data (as well as the solutions) are radially symmetric. The general case $\gamma \geqslant 1$, $N = 3$ for radially symmetric data was solved only recently by Jiang and Zhang [55].

6.3. *Time-periodic solutions*

Similarly as above, we consider the system (5.3)–(5.5) driven by a volume force $\mathbf{f}$ which is periodic in time, i.e., $\mathbf{f}$ is a bounded measurable vector function on $R \times \Omega$ satisfying

$$\mathbf{f}(t+\omega, x) = \mathbf{f}(t, x) \quad \text{for a.a. } t \in R,\ x \in \Omega$$

for a certain period $\omega > 0$. We are interested in the existence of a finite energy weak solution ϱ, $\mathbf{u}$ enjoying the same property, i.e.,

$$\varrho(t+\omega) = \varrho(t), \qquad (\varrho\mathbf{u})(t+\omega) = (\varrho\mathbf{u})(t) \quad \text{for all } t \in R$$

and such that

$$\int_\Omega \varrho \,\mathrm{d}x = m,$$

where m is a given positive total mass.

There are three main obstacles making this problem rather delicate. Given the existence results for the initial-boundary value problem presented above, only weak solutions are available, for which the question of uniqueness is highly nontrivial and far from being solved. This excludes all the so-called indirect methods based on fixed-point arguments for the corresponding period map. While the former difficulty might seem only technical, there is another feature of the problem, mentioned already in Section 5.4 namely, there is no "solution operator" or "period map" which would be compact due to possible time propagation of oscillations in the density. Last but not the least, fixing the total mass m, we have to look for solutions lying on a sphere in the space L^1 which excludes the possibility of using any fixed-point technique in a direct fashion.

In the light of the above arguments, the only possibility to get positive results is to work directly in the space of periodic solutions that means to consider a genuine boundary-value problem for the evolutionary system (5.3), (5.4). This approach has been used in [31] to prove the existence of the time periodic solutions to (5.3), (5.4) on a cube in R^3 complemented by the no-stick boundary conditions (1.12). Combining the method of [31] with the existence theory [32] one can prove the following result.

THEOREM 6.3. *Let $\Omega \subset R^N$, $N = 2, 3$, be a bounded regular domain. Consider the problem* (5.3)–(5.5) *where p is given by the isentropic constitutive law* (5.2) *with*

$$\gamma > 5/3 \quad \text{if } N = 3, \qquad \gamma > 1 \quad \text{for } N = 2,$$

and $\mathbf{f}$ is a bounded measurable function on $R \times \Omega$ such that

$$\mathbf{f}(t+\omega, x) = \mathbf{f}(t, x) \quad \text{for a.a. } t \in R,\ x \in \Omega \text{ and a certain } \omega > 0.$$

Then, given $m > 0$*, there exists a finite energy weak solution* ϱ, $\mathbf{u}$ *of* (5.3)–(5.5) *on* $R \times \Omega$ *such that*

$$\varrho(t+\omega) = \varrho(t), \qquad (\varrho\mathbf{u})(t+\omega) = (\varrho\mathbf{u})(t) \quad \text{for all } t \in R$$

and

$$\int_{\Omega} \varrho \, \mathrm{d}x = m.$$

The condition $\gamma > 5/3$ in the three-dimensional case seems rather strange compared with $\gamma > 3/2$ required for solving the initial-value problem. This is related to the problem of ultimate boundedness or resonance phenomena for global in time solutions. We will discuss this interesting topic in the next section.

6.4. *Counter-examples to global existence*

It is not clear to which extent the hypothesis $\gamma > N/2$ is really necessary for global existence results. Several attempts have been made to show that the barotropic model does not admit globally defined strong or even weak solutions but the results are still not very convincing in either positive or negative sense.

Following the method of Vaigant [106], Desjardins [19, Proposition 1] studied the integrability properties of the density ϱ in the system (5.3)–(5.5).

PROPOSITION 6.1. *Let* $\Omega = B(1) \subset R^3$ *be a unit ball and let* p *satisfy* (5.2) *with* $1 < \gamma < 3$. *Let*

$$q > \frac{11\gamma - 2}{6}.$$

Then there exist $\mathbf{f} \in L^1(0, T; L^{\frac{2\gamma}{\gamma-1}}(\Omega))$ *and a globally defined weak solution* ϱ, $\mathbf{u}$ *of* (5.3)–(5.5) *such that*

$$\int_0^T \int_{B(1/2)} \left|\varrho(t,x)\right|^q \mathrm{d}x \, \mathrm{d}t = \infty.$$

The weakness of this result stems from the necessity to use the forcing term $\mathbf{f}$ which is singular at $t = T$. It is still an open problem whether or not the uniform upper bounds on the density can be obtained independently of the choice of γ.

6.5. *Possible generalization*

We shall comment shortly on possible improvements of Theorem 6.1 lying in the scope of the present theory.

To begin with, Theorem 6.1 still holds when Ω is a general (not necessarily bounded) domain with compact boundary on which the no-slip boundary conditions for the velocity are prescribed. As far as the other boundary conditions discussed in Section 1.4 are concerned, the possibility to show positive existence results seems to be closely related to the question of the boundary estimates of the pressure discussed in Section 5.2.

Similarly, the hypothesis that $\mathbf{f}$ is bounded can be replaced by a more general condition $\mathbf{f} \in L^1(I; L^{\frac{2\gamma}{\gamma-1}}(\Omega))$.

Other possibilities and suggestions are discussed by Lions [62].

7. Barotropic flows: asymptotic properties

Similarly as in the preceding section, we focus on the system (5.2)–(5.5) considered on a bounded regular domain $\Omega \subset R^N$, $N = 2, 3$. We shall assume that the driving force f is a bounded measurable function defined, for simplicity, for all $t \in R$, $x \in \Omega$ such that

$$\left|\mathbf{f}(t, x)\right| \leqslant F \quad \text{for a.a. } t \in R,\ x \in \Omega. \tag{7.1}$$

In accordance with Section 5, the total energy defined as

$$\mathcal{E}[\varrho, \varrho\mathbf{u}](t) = \int_{\varrho(t)>0} \frac{1}{2}\frac{|\varrho\mathbf{u}|^2}{\varrho}(t) + \frac{a}{\gamma-1}\varrho^{\gamma}(t)\,\mathrm{d}x$$

is a lower-semicontinuous function of t.

7.1. *Bounded absorbing balls and stationary solutions*

We shall address the problem of ultimate boundedness of global in time finite energy weak solutions, the existence of which is guaranteed by Theorem 6.2. We shall see that the total energy $\mathcal{E}$ is the right quantity to play the role of a "norm" in these considerations. If the driving force $\mathbf{f}$ is uniformly bounded as in (7.1), the "dynamical system" generated by the finite energy weak solutions of the problem (5.3)–(5.5) is ultimately bounded or dissipative in the sense of Levinson with respect to the energy "norm" provided that the adiabatic constant satisfies $\gamma > 1$ for $N = 2$ and $\gamma > 5/3$ if $N = 3$. Specifically, we report the following result (see [38, Theorem 1.1]), the proof of which is based on the pressure estimates obtained in Proposition 5.2:

THEOREM 7.1. *Let $\Omega \subset R^N$, $N = 2, 3$, be a bounded Lipschitz domain and $I \subset R$ an interval such that* $\inf\{I\} > -\infty$. *Consider the system* (5.3)–(5.5) *with the isentropic pressure p given by* (5.2) *with*

$$\gamma > 1 \quad \textit{if } N = 2, \qquad \gamma > 5/3 \quad \textit{for } N = 3, \tag{7.2}$$

and $\mathbf{f}$ *satisfying* (7.1).

Then there exists a constant $\mathcal{E}_\infty$, depending solely on the amplitude of the driving force F and the total mass m, with the following property:
Given $\mathcal{E}_0$, there exists a time $T = T(\mathcal{E}_0)$ such that

$$\mathcal{E}\big[\varrho, (\varrho \mathbf{u})\big](t) \leqslant \mathcal{E}_\infty \quad \textit{for all } t \in I,\ t > T + \inf\{I\}$$

for any ϱ, $\mathbf{u}$ – a finite energy weak solution of the problem (5.3)–(5.5) *– satisfying*

$$\limsup_{t\to\inf\{I\}+} \mathcal{E}[\varrho, \mathbf{u}](t) \leqslant \mathcal{E}_0, \qquad \int_\Omega \varrho \, \mathrm{d}x = m.$$

It seems interesting to compare Theorem 7.1 with the result of Lions [62, Theorem 6.7] on the existence of stationary solutions of (5.3)–(5.5) to shed some light on the role of the hypothesis (7.2).

THEOREM 7.2. *Let $\Omega \subset R^N$, $N = 2, 3$, be a bounded regular domain, $\mathbf{f} = \mathbf{f}(x)$ a function belonging to $L^\infty(\Omega)$, and $m > 0$. Assume $p = p(\varrho)$ is given by* (5.2) *with γ satisfying* (7.2).

Then there exists a pair of functions $\varrho = \varrho(x) \in L^p(\Omega)$, $p > \gamma$, $\mathbf{u} = \mathbf{u}(x) \in W_0^{1,2}(\Omega)$ solving the stationary problem

$$\mathrm{div}(\varrho \mathbf{u}) = 0,$$

$$\mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + a\nabla\varrho^\gamma = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla \,\mathrm{div}\, \mathbf{u} + \varrho\mathbf{f},$$

$$\int_\Omega \varrho \, \mathrm{d}x = m$$

in $\mathcal{D}'(\Omega)$.

As we will see later, Theorem 7.2 can be deduced from Proposition 5.4, Theorem 6.3, and Theorem 7.1.

The property stated in Theorem 7.1 is evidence of the dissipative nature of the system (5.3), (5.4). In finite-dimensional setting, J.E. Billoti and J.P. LaSalle proposed it as a definition of dissipativity. Unfortunately, however, some difficulties inherent to infinite-dimensional dynamical systems make it, in that case, less appropriate.

7.2. *Complete bounded trajectories*

We suppose that the driving force $\mathbf{f}$ belongs to $\mathcal{F}$ – a bounded subset of $L^\infty_{\mathrm{loc}}(R; L^\infty(\Omega))$. To bypass the possible problem of non-uniqueness of finite energy weak solutions, we introduce a quantity $U(t_0, t)$ playing the role of the *evolution operator* related to the problem (5.3)–(5.5).

$$U[\mathcal{E}_0,\mathcal{F}](t_0,t)=\Big\{\big[\varrho(t),(\varrho\mathbf{u})(t)\big]\mid\varrho,\ \mathbf{u}\text{ is a finite energy weak solution of the problem (5.3)–(5.5) defined on an open interval } I,\ (t_0,t]\subset I,\ \text{with } \mathbf{f}\in\mathcal{F} \text{ and such that } \limsup_{t\to t_0+}\mathcal{E}[\varrho,\mathbf{u}](t)\leqslant\mathcal{E}_0\Big\}.$$

We start with the concept of the so-called short trajectory in the spirit of Málek and Nečas [67]:

$$U^s[\mathcal{E}_0,\mathcal{F}](t_0,t)=\Big\{\big[\varrho(t+\tau),(\varrho\mathbf{u})(t+\tau)\big],\ \tau\in[0,1]\mid\varrho,\mathbf{u}\text{ is a finite energy weak solution of the problem (5.3)–(5.5) on an open interval } I,\ (t_0,t+1]\subset I,\ \text{with } \mathbf{f}\in\mathcal{F},\quad\text{and such that } \limsup_{t\to t_0}\mathcal{E}(t)\leqslant\mathcal{E}_0\Big\}.$$

The following result can be viewed as a corollary of Proposition 5.4 and Theorem 7.1 (cf. [37, Theorem 1.1] or [27, Proposition 10]).

PROPOSITION 7.1. *Let $\Omega\subset R^N$, $N=2,3$, be a bounded domain with Lipschitz boundary. Let the pressure p be given by* (5.2) *with*

$$\gamma>1\quad\text{for } N=2,\qquad\gamma>5/3\quad\text{if } N=3.$$

Let $\mathcal{F}$ be bounded in $L^\infty(R\times\Omega)$. Consider a sequence $[\varrho_n,(\varrho_n\mathbf{u}_n)]\in U^s[\mathcal{E}_0,\mathcal{F}](a,t_n)$ for a certain $t_n\to\infty$.

Then there is a subsequence (not relabeled) such that

$$\varrho_n\to\varrho\quad\text{in } L^\gamma\big((0,1)\times\Omega\big)\text{ and in } C\big([0,1];L^\alpha(\Omega)\big)\text{ for } 1\leqslant\alpha<\gamma,$$

$$\varrho_n\mathbf{u}_n\to(\varrho\mathbf{u})\quad\text{in } L^p((0,1)\times\Omega)\text{ and in } C([0,1];L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega))\text{ for any } 1\leqslant p<\tfrac{2\gamma}{\gamma+1},\text{ and}$$

$$\mathcal{E}\big[\varrho_n,(\varrho_n\mathbf{u}_n)\big]\to\mathcal{E}\big[\varrho,(\varrho\mathbf{u})\big]\quad\text{in } L^1(0,1),$$

where ϱ, $\mathbf{u}$ is a finite energy weak solution of the problem (5.3)–(5.5) *defined on the whole real line $I=R$ such that $\mathcal{E}\in L^\infty(R)$ and with $\mathbf{f}\in\mathcal{F}^+$ where*

$$\mathcal{F}^+=\Big\{\mathbf{f}\mid\mathbf{f}=\lim_{\tau_n\to\infty}\mathbf{h}_n(\cdot+\tau_n)\text{ weak star in } L^\infty(R\times\Omega)\ \text{for a certain } \mathbf{h}_n\in\mathcal{F}\text{ and } \tau_n\to\infty\Big\}.$$

Proposition 7.1 shows the importance of the complete bounded trajectories, i.e., the finite energy weak solutions defined on $I = R$ whose total energy $\mathcal{E}$ is uniformly bounded on R. Let us define

$$\begin{aligned}\mathcal{A}^s[\mathcal{F}] = \big\{ \big[\varrho(\tau), (\varrho\mathbf{u})(\tau) \big],\ \tau \in [0,1] \mid \varrho, \mathbf{u} \text{ is a finite energy weak solution} \\ \text{of the problem (5.3)–(5.5) on the interval } I = R, \\ \text{with } \mathbf{f} \in \mathcal{F}^+ \text{ and } \mathcal{E}\big[\varrho, (\varrho\mathbf{u})\big] \in L^\infty(R) \big\}.\end{aligned}$$

The next statement is a straighforward consequence of Proposition 7.1 (see also [28, Theorem 3.1]).

THEOREM 7.3. *Let $\Omega \subset R^N$ be a bounded Lipschitz domain. Let p be given by* (5.2) *with*

$$\gamma > 1 \quad \textit{for } N = 2, \qquad \gamma > 5/3 \quad \textit{if } N = 3.$$

Let $\mathcal{F}$ be a bounded subset of $L^\infty(R \times \Omega)$.

Then the set $\mathcal{A}^s[\mathcal{F}]$ is compact in $L^\gamma((0,1) \times \Omega) \times [L^p((0,1) \times \Omega)]^3$ and

$$\begin{aligned}\sup_{[\varrho,\varrho\mathbf{u}] \in U[\mathcal{E}_0,\mathcal{F}](t_0,t)} \Big[\inf_{[\bar\varrho,\bar\varrho\bar{\mathbf{u}}] \in \mathcal{A}^s[\mathcal{F}]} \big(\|\varrho - \bar\varrho\|_{L^\gamma((0,1)\times\Omega)} \\ + \big\| (\varrho\mathbf{u}) - (\bar\varrho\bar{\mathbf{u}}) \big\|_{L^p((0,1)\times\Omega)} \big) \Big] \to 0 \quad \textit{as } t \to \infty\end{aligned}$$

for any $1 \leqslant p < 2\gamma/(\gamma+1)$.

Theorem 7.3 says that the set $\mathcal{A}^s[\mathcal{F}]$ is a global attractor on the space of “short” trajectories. This is a result in the spirit of Málek and Nečas [67] or Sell [89].

In particular, the set $\mathcal{A}^s[\mathcal{F}]$ is compact non-empty provided $\mathcal{F}$ is non-empty. Consider the special case when $\mathbf{f} = \mathbf{f}(x)$ is a driving force independent of time. Accordingly, we can take

$$\mathcal{F} = \mathcal{F}^+ = \{\mathbf{f}\}.$$

By virtue of Theorem 6.3, the problem (5.3)–(5.5) possesses a time-periodic solution ϱ_n, $\mathbf{u}_n$ for any period $\omega_n = 2^{-n}$ such that

$$\int_\Omega \varrho_n \, \mathrm{d}x = m.$$

Moreover, Theorem 7.1 implies that the restriction $\varrho_n, \varrho_n\mathbf{u}_n$ to the time interval $[0,1]$ belongs to $\mathcal{A}^s$. As $\mathcal{A}^s$ is compact, the sequence ϱ_n, $\mathbf{u}_n$ has an accummulation point which is a complete global solution of (5.3)–(5.5). Moreover, this solution is clearly independent of t, i.e., it is a stationary solution of a given total mass m. In other words, we have proved Theorem 7.2.

7.3. *Potential flows*

We shall examine the flows driven by a potential force, i.e., we assume

$$\mathbf{f} = \mathbf{f}(x) = \nabla F(x),$$

where F is a Lipschitz continuous function.

In this case, the term on the right-hand side of the energy inequality (5.6) can be rewritten as

$$\int_\Omega (\varrho \mathbf{u}) \cdot \nabla F \, \mathrm{d}x = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}, \quad \text{where } \mathcal{H}(t) = \int_\Omega \varrho F \, \mathrm{d}x,$$

and, consequently, (5.6) reads as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{E}(t) - \mathcal{H}(t)\big) + \int_\Omega \mu |\nabla \mathbf{u}|^2 + (\lambda + 2\mu)|\operatorname{div} \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant 0. \tag{7.3}$$

We denote

$$\mathcal{E}\mathcal{H}_\infty = \operatorname{ess} \lim_{t \to \infty} \big[\mathcal{E}(t) - \mathcal{H}(t)\big].$$

By virtue of (7.3) and the Poincaré inequality, the integral

$$\int_1^\infty \|\mathbf{u}\|^2_{W_0^{1,2}(\Omega)} \, \mathrm{d}t \text{ is convergent,} \tag{7.4}$$

in particular,

$$\lim_{T \to \infty} \int_T^{T+1} \mathcal{E}_{\mathrm{kin}}(t) \, \mathrm{d}t = 0, \quad \mathcal{E}_{\mathrm{kin}} = \frac{1}{2} \int_\Omega \varrho |\mathbf{u}|^2 \, \mathrm{d}x,$$

and

$$\lim_{T \to \infty} \int_T^{T+1} \int_\Omega \frac{a}{\gamma - 1} \varrho^\gamma - \varrho F \, \mathrm{d}x \, \mathrm{d}t = \mathcal{E}\mathcal{H}_\infty. \tag{7.5}$$

Similarly as in Proposition 7.1, one can show that any sequence $t_n \to \infty$ contains a subsequence such that

$$\int_{t_n}^{t_n+1} \|\varrho(t) - \varrho_s\|_{L^\gamma(\Omega)} \, \mathrm{d}t \to 0,$$

where, in view of (7.4), (7.5), ϱ_s is a solution of the stationary problem

$$\begin{aligned} &a \nabla \varrho_s^\gamma = \varrho_s \nabla F \quad \text{in } \Omega, \qquad \int_\Omega \varrho_s \, \mathrm{d}x = m, \\ &\int_\Omega \frac{a}{\gamma - 1} \varrho_s^\gamma - \varrho_s F \, \mathrm{d}x = \mathcal{E}\mathcal{H}_\infty. \end{aligned} \tag{7.6}$$

Consequently, it is of interest to study the structure of the set of the static solutions, i.e., the solutions of the problem (7.6); in particular, whether or not they form a discrete set. If this is the case, any finite energy weak solution of (5.3)–(5.5) is convergent to a static state. A partial answer was obtained in the case of potentials with at most two "peaks" ([36, Theorem 1.1] and [33, Theorem 1.2]).

THEOREM 7.4. *Let $\Omega \subset R^N$ be an arbitrary domain.*

(i) *Assume F is locally Lipschitz continuous on Ω and such that all the upper level sets*

$$[F > k] = \{x \in \Omega \mid F(x) > k\}$$

are connected in Ω for any k.

Then given $m > 0$, the problem (7.6) possesses at most one nonnegative solution ϱ_s.

(ii) *If F is locally Lipschitz continuous and Ω can be decomposed as*

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset,$$

where Ω_i are two subdomains (one of them possibly empty) so that

$$[F > k] \cap \Omega_i \quad \text{is connected in } \Omega_i \text{ for } i = 1, 2 \text{ for any } k \in R, \tag{7.7}$$

then, given m, $\mathcal{EH}_\infty$, the problem (7.6) admits at most two distinct non-negative solutions.

Making use of Theorem 7.4, one can show the following result on stabilization of global solutions for potential flows (cf. [34, Theorem 1.1], [27, Theorem 15]).

THEOREM 7.5. *Let $\Omega \subset R^N$, $N = 2, 3$, be a bounded Lipschitz domain. Let the pressure p satisfy the constitutive relation (5.2) with $\gamma > N/2$. Let $\mathbf{f} = \mathbf{f}(x) = \nabla F(x)$ where F is globally Lipschitz potential on Ω. Moreover, assume that Ω can be decomposed as in Theorem 7.4 so that (7.7) holds.*

Then for any finite energy weak solution ϱ, $\mathbf{u}$ of the problem (5.3)–(5.5) defined on a time interval $I = (t_0, \infty)$, there exists a solution ϱ_s of the stationary problem (7.6) such that

$$\varrho(t) \to \varrho_s \quad \text{strongly in } L^\gamma(\Omega) \text{ as } t \to \infty,$$

$$\int_\Omega E_{\text{kin}}(t)\,\mathrm{d}x = \frac{1}{2}\int_{\varrho(t)>0} \frac{|\varrho\mathbf{u}|^2}{\varrho}(t)\,\mathrm{d}x \ \to 0 \quad \text{as } t \to \infty.$$

The conclusion of Theorem 7.5 still holds if Ω is a general (not necessarily bounded) domain with compact boundary provided F satisfies the stronger hypothesis of Theorem 7.4, namely, all upper level sets $[F > k]$ must be connected. Similar problems on the exterior of an open ball and for radially symmetric solutions were investigated by Matušů-Nečasová et al. [76]. Related results can be found in Novotný and Straškraba [83,84].

It is an interesting open problem if the conclusion of Theorem 7.5 still holds when the hypothesis on the upper level sets of F is relaxed. If $\Omega \subset R^N$ is a bounded domain and the potential F nontrivial (nonconstant), there always exists an m – the total mass – small in comparison with F such that the solutions of the static problem (7.6) contain vacuum zones (cf. [34, Section 5]). Thus for *any* nonconstant F the global solutions approach rest states with vacuum regions as time goes to infinity. One should note in this context there are many formal results on convergence of isentropic flows to a stationary state under various hypotheses including uniform (in time) boundedness of the density away from zero (see, e.g., Padula [86]). As we have just observed, this can be rigorously verified only for solutions representing small perturbations of strictly positive rest states.

7.4. *Highly oscillating external forces*

There seems to be a common belief that highly oscillating driving forces of zero integral mean do not influence the long-time dynamics of dissipative systems. Averaging a function over a short time interval should be considered analogous to making a macroscopic measurement in a physical experiment. The result of such an experiment being close to zero, the effect on the solutions to a sufficiently robust dynamical systems, if any, should be negligible at least in the long run. From the mathematical point of view, these ideas have been made precise by Chepyzhov and Vishik [10] dealing with trajectory attractors of evolution equations. They showed that the trajectory attractors of certain dissipative dynamical systems perturbed by a highly oscillating forcing term are the same as for the unperturbed system. Their results apply to a vast set of equations including the damped wave equations and the Navier–Stokes equations for incompressible fluids. Our goal now is to present comparable results for the problem of isentropic compressible flows dynamics.

Highly oscillating sequences converge in the weak topology, i.e., the topology of convergence of integral means. Consider a ball B_G of radius G centered at zero in the space $L^\infty((0,1)\times\Omega)$. The weak-star topology on B_G is metrizable and we denote the corresponding metric d_G. We report the following result (see [30, Theorem 1.2]).

THEOREM 7.6. *Assume $\Omega \subset R^N$, $N = 2, 3$ is a bounded Lipschitz domain. Consider the system* (5.3)–(5.5) *where the pressure p is given by* (5.2) *with*

$$\gamma > 1 \quad \textit{for } N = 2, \qquad \gamma > 5/3 \quad \textit{if } N = 3,$$

and

$$\mathbf{f}(t,x) = \nabla F(x) + \mathbf{g}(t,x),$$

where F is globally Lipschitz continuous and such that the upper level sets $[F > k]$ are connected for any k.

Then given $G > 0$, $\varepsilon > 0$ there exists $\delta = \delta(G,\varepsilon) > 0$ such that

$$\limsup_{t\to\infty}\big[\|\varrho(t) - \varrho_s\|_{L^\gamma(\Omega)} + \|\varrho\mathbf{u}(t)\|_{L^1(\Omega)}\big] < \varepsilon$$

for any finite energy weak solution ϱ, $\mathbf{u}$ *of the problem* (5.3)–(5.5) *provided*

$$\limsup_{t\to\infty} \|\mathbf{g}(t)\|_{L^\infty((t,\infty)\times\Omega)} < G,$$
$$\limsup_{t\to\infty} d_G\big[\mathbf{g}(t+s)|_{s\in[0,1]}, 0\big] < \delta.$$

Here ϱ_s *is the unique solution of the stationary problem* (7.6).

7.5. *Attractors*

For a general dynamical system a set $\mathcal{A}$ is called a *global attractor* if it is compact, attracting all trajectories, and minimal in the sense of inclusion in the class of sets having the first two properties. The theory of attractors for *incompressible* flows is well developed. We refer the reader to the monographs of Babin and Vishik [5], Hale [44], and Temam [104] for this interesting subject. A global or universal attractor describes all possible dynamics of a given system, and, as an aspect of dissipativity, the attractor usually has a finite fractal dimension.

There seems to be at least one essential problem to develop a sensible *dynamical systems theory* for compressible fluids, namely, the finite energy weak solutions we deal with are not known to be uniquely determined by the initial data. On the other hand, the notion of global attractor itself does not require uniqueness or even the existence of a "solution semigroup" and plausible results in this respect can be obtained.

Let

$$\mathcal{A}[\mathcal{F}] = \big\{\big[\varrho(0), (\varrho\mathbf{u})(0)\big] \mid \varrho, \mathbf{u} \text{ is a finite energy weak solution}$$
$$\text{of the problem (5.3)–(5.5) on } I = R \text{ with } \mathbf{f} \in \mathcal{F}^+ \text{ and } \mathcal{E} \in L^\infty(R)\big\}.$$

Roughly speaking, the set $\mathcal{A}$ contains all global and globally bounded trajectories where global means defined on the whole time axis R.

The next statement shows that $\mathcal{A}[\mathcal{F}]$ is a global attractor in the sense of Foias and Temam [40] (cf. [28, Theorem 4.1]).

THEOREM 7.7. *Let* $\Omega \subset R^N$, $N = 2, 3$, *be a bounded Lipschitz domain, and let* p *be given by* (5.2) *with*

$$\gamma > 1 \quad \textit{if } N = 2, \qquad \gamma > 5/3 \quad \textit{for } N = 3.$$

Let $\mathcal{F}$ *be a bounded subset of* $L^\infty(R\times\Omega)$.

Then $\mathcal{A}[\mathcal{F}]$ *is compact in* $L^\alpha(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega)$ *and*

$$\sup_{[\varrho,\varrho\mathbf{u}]\in U[\mathcal{E}_0,\mathcal{F}](t_0,t)} \left[\inf_{[\bar\varrho,\bar\varrho\bar{\mathbf{u}}]\in\mathcal{A}[\mathcal{F}]} \left(\|\varrho-\bar\varrho\|_{L^\alpha(\Omega)} + \left|\int_\Omega (\varrho\mathbf{u}-\bar\varrho\bar{\mathbf{u}})\cdot\phi\,\mathrm{d}x\right|\right)\right] \to 0$$
$$\text{as } t\to\infty$$

for any $1 \leqslant \alpha < \gamma$ *and any* $\phi \in [L^{\frac{2\gamma}{\gamma-1}}(\Omega)]^3$.

The apparent shortcomming of this result is that $\mathcal{A}$ is only a "weak" attractor with respect to the momentum component. Pursuing the idea of Ball [6], one can show a stronger result on condition that some additional smoothness of $\mathcal{A}$ is known (see [27, Theorem 17]).

THEOREM 7.8. *In addition to the hypotheses of Theorem* 7.7, *assume the total energy* $\mathcal{E}$ *defined by* (5.9) *and considered as a function the density* ϱ *and the momenta* $\varrho\mathbf{u}$ *is* (*sequentially*) *continuous on* $\mathcal{A}[\mathcal{F}]$, *specifically, for any sequence*

$$[\varrho_n, \mathbf{q}_n] \in \mathcal{A}[\mathcal{F}] \quad \text{such that } \varrho_n \to \varrho \text{ in } L^1(\Omega),\ \mathbf{q}_n \to \mathbf{q} \text{ weakly in } L^1(\Omega)$$

one requires

$$\mathcal{E}[\varrho_n, \mathbf{q}_n] \to \mathcal{E}[\varrho, \mathbf{q}].$$

Then

$$\sup_{[\varrho,\varrho\mathbf{u}]\in U[\mathcal{E}_0,\mathcal{F}](t_0,t)} \Big[\inf_{[\bar\varrho,\bar\varrho\bar{\mathbf{u}}]\in\mathcal{A}[\mathcal{F}]} \big(\|\varrho-\bar\varrho\|_{L^\gamma(\Omega)} + \|\varrho\mathbf{u}-\bar\varrho\bar{\mathbf{u}}\|_{L^1(\Omega)}\big)\Big] \to 0$$

as $t \to \infty$.

7.6. *Bibliographical remarks*

The existence of global attractors for the problem (5.3)–(5.5) with $\gamma = 1$ and $N = 1$ was studied by Hoff and Ziane [50,51]. In this case, any forcing term f is of potential type so the only situation which is not covered by Section 7.3 is the case when f is time dependent.

Similar results for the full system (1.1)–(1.3), still in one space dimension, were obtained recently by Zheng and Qin [112].

8. Compressible–incompressible limits

It is well-accepted in fluid mechanics that one can derive formally incompressible models as Navier–Stokes equations from compressible ones. Such a situation can be expected when letting the Mach number go to zero in the isentropic compressible Navier–Stokes equations. Following Lions and Masmoudi [64] we consider a system

$$\frac{\partial\varrho_\varepsilon}{\partial t} + \operatorname{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon) = 0, \tag{8.1}$$

$$\frac{\partial\varrho_\varepsilon\mathbf{u}_\varepsilon}{\partial t} + \operatorname{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) + \frac{a}{\varepsilon^2}\nabla\varrho_\varepsilon^\gamma = \mu_\varepsilon\Delta\mathbf{u}_\varepsilon + (\lambda_\varepsilon+\mu_\varepsilon)\nabla\operatorname{div}\mathbf{u}_\varepsilon \tag{8.2}$$

complemented by the initial conditions

$$\varrho_\varepsilon(0) = \varrho_\varepsilon^0 \geqslant 0, \qquad (\varrho_\varepsilon\mathbf{u}_\varepsilon)(0) = \mathbf{q}_\varepsilon \tag{8.3}$$

satisfying (6.7). We shall always assume

$$\mu_\varepsilon \to \mu > 0, \qquad \lambda_\varepsilon \to \lambda > -\mu \quad \text{as } \varepsilon \to 0.$$

8.1. *The spatially periodic case*

In addition to the above hypotheses, assume the initial data are spatially periodic as in (6.3). Moreover, let

$$\frac{\mathbf{q}_\varepsilon}{\sqrt{\varrho_\varepsilon^0}} \to \mathbf{U}_0 \quad \text{weakly in } L^2_{\text{per}}(R^N) \text{ as } \varepsilon \to 0, \tag{8.4}$$

and

$$\int_0^{\omega_1} \cdots \int_0^{\omega_N} \frac{|\mathbf{q}_\varepsilon|^2}{\varrho_\varepsilon^0} + \frac{1}{\varepsilon^2}\big[(\varrho_\varepsilon^0)^\gamma - \gamma \varrho_\varepsilon^0 (m_\varepsilon^0)^{\gamma-1} + (\gamma-1)(m_\varepsilon^0)^\gamma\big]\,\mathrm{d}x \leqslant c \tag{8.5}$$

where

$$m_\varepsilon^0 = \left(\prod_{i=1}^N \omega_i\right)^{-1} \int_0^{\omega_1} \cdots \int_0^{\omega_N} \varrho_\varepsilon^0 \,\mathrm{d}x \to 1 \quad \text{as } \varepsilon \to 0$$

independently of ε.

Let us denote, as usual, the total energy

$$\mathcal{E}\big[\varrho_\varepsilon, (\varrho_\varepsilon \mathbf{u}_\varepsilon)\big] = \int_{\{\varrho_\varepsilon > 0\}} \frac{1}{2}\frac{|\varrho_\varepsilon \mathbf{u}_\varepsilon|^2}{\varrho_\varepsilon} + \frac{a}{\varepsilon^2(\gamma-1)}(\varrho_\varepsilon)^\gamma \,\mathrm{d}x.$$

The following result is due to Lions and Masmoudi [64]:

THEOREM 8.1. *Assume $\gamma > N/2$. Let ϱ_ε, $\mathbf{u}_\varepsilon$ be a (spatially periodic) finite energy weak solution of the problem* (8.1)–(8.3) *on the time interval* $(0, \infty)$ *where the data satisfy* (8.4), (8.5). *Moreover, let*

$$\operatorname*{ess\,sup}_{t \to 0+} \mathcal{E}\big[\varrho_\varepsilon, (\varrho_\varepsilon \mathbf{u}_\varepsilon)\big] \leqslant \int_0^{\omega_1} \cdots \int_0^{\omega_N} \frac{1}{2}\frac{|\mathbf{q}_\varepsilon|^2}{\varrho_\varepsilon^0} + \frac{a}{\varepsilon^2(\gamma-1)}\big(\varrho_\varepsilon^0\big)^\gamma \,\mathrm{d}x.$$

Then

$$\varrho_\varepsilon \to 1 \quad \textit{in } C\big([0,T]; L^\gamma_{\text{per}}\big(R^2\big)\big)$$

and $\mathbf{u}_\varepsilon$ is bounded in $L^2(0, T; W^{1,2}_{\text{per}}(\Omega))$ for arbitrary $T > 0$.

Moreover, passing to a subsequence as the case may be we have

$$\mathbf{u}_\varepsilon \to \mathbf{U} \quad \textit{weakly in } L^2\big(0, T; W^{1,2}_{\text{per}}\big(R^2\big)\big),$$

where $\mathbf{U}$ *solves the incompressible Navier–Stokes equations*

$$\partial_t \mathbf{U} + \operatorname{div}(\mathbf{U} \otimes \mathbf{U}) = \mu \Delta \mathbf{U} + \nabla P, \qquad \operatorname{div} \mathbf{U} = 0 \tag{8.6}$$

with the initial condition $\mathbf{U}(0) = \mathcal{P}\mathbf{U}_0$ *where* $\mathcal{P}$ *is the projection on the space of divergence-free functions.*

8.2. *Dirichlet boundary conditions*

Now we focus on the system (8.1)–(8.3) posed on a bounded domain $\Omega \subset R^N$ and complemented by the no-slip boundary conditions for the velocity:

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = 0. \tag{8.7}$$

Consider the following (overdetermined) problem:

$$-\nabla \Phi = \nu \Phi \quad \text{in } \Omega, \qquad \nabla \Phi \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \Phi \text{ constant on } \partial\Omega. \tag{8.8}$$

A solution of (8.8) is trivial if $\nu = 0$ and Φ is a constant. The domain Ω will be said to satisfy condition (H) if all solutions of (8.8) are trivial.

The following result was proved by Desjardins et al. [20]:

THEOREM 8.2. *Let* $\Omega \subset R^N$, $N = 2, 3$, *be a bounded regular domain. In addition to the hypotheses of Theorem* 8.1, *assume that* $\mathbf{u}_\varepsilon$ *satisfies the no-slip condition* (8.7).

Then ϱ_ε *tends to* 1 *strongly in* $C([0, T]; L^\gamma(\Omega))$ *and, passing to a subsequence if necessary,*

$$\mathbf{u}_\varepsilon \to \mathbf{U} \quad \textit{weakly in } L^2\big((0, T) \times \Omega\big)$$

for all $T > 0$ *and the convergence is strong if* Ω *satisfies condition* (H). *In addition,* $\mathbf{U}$ *satisfies the incompressible Navier–Stokes system* (8.6) *complemented by the no-slip boundary conditions on* $\partial\Omega$ *and with* $\mathbf{U}(0) = \mathcal{P}\mathbf{U}_0$.

8.3. *The case* $\gamma_n \to \infty$

Let us consider the isentropic system in the case when $\gamma = \gamma_n \to \infty$. We follow the presentation of Lions and Masmoudi [65].

Let $\Omega \subset R^3$ be a bounded regular domain. Consider the system

$$\partial_t \varrho_n + \operatorname{div}(\varrho_n \mathbf{u}_n) = 0, \tag{8.9}$$

$$\partial_t(\varrho_n \mathbf{u}_n) + \operatorname{div}(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + a\nabla\varrho_n^{\gamma_n} = \mu\Delta\mathbf{u}_n + (\lambda+\mu)\nabla \operatorname{div}\mathbf{u}_n \tag{8.10}$$

with the no-slip boundary conditions for the velocity

$$\mathbf{u}_n|_{\partial\Omega} = 0 \tag{8.11}$$

and complemented by the initial conditions

$$\varrho_n(0) = \varrho_n^0 \geqslant 0, \qquad (\varrho_n \mathbf{u}_n)(0) = \mathbf{q}_n, \tag{8.12}$$

where

$$\left|\varrho_n^0\right|^{\gamma_n}_{L^{\gamma_n}(\Omega)} \leqslant c\gamma_n, \quad \varrho_n \text{ bounded in } L^1(\Omega), \ \frac{|\mathbf{q}_n|^2}{\varrho_0} \text{ bounded in } L^1(\Omega), \tag{8.13}$$

independently of n.

We are interested in the limit of the sequence ϱ_n, $\mathbf{u}_n$ of finite energy weak solutions of the problem (8.9)–(8.12) when $\gamma_n \to \infty$. To this end, let us first formulate the limit problem:

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0, \quad 0 \leqslant \varrho \leqslant 1, \tag{8.14}$$

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla\mathcal{P} = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u}, \tag{8.15}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{a.a. on the set } \{\varrho = 1\}, \tag{8.16}$$

$$\mathcal{P} = 0 \quad \text{a.a. on } \{\varrho < 1\}, \qquad \mathcal{P} \geqslant 0 \quad \text{a.a. on } \{\varrho = 1\}. \tag{8.17}$$

The following result is due to Lions and Masmoudi [65].

THEOREM 8.3. *Let $\Omega \subset R^3$ be a bounded regular domain. Let ϱ_n, $\mathbf{u}_n$ be a sequence of finite energy weak solutions of the problem* (8.9)–(8.12) *on* $(0,T)\times\Omega$ *where the data satisfy* (8.13). *Let ϱ_n^0 converge weakly to some ϱ^0 and $\mathbf{q}_n$ converge weakly to $\mathbf{q}$.*

Then ϱ_n, $\mathbf{u}_n$ contain subsequences such that

$$[\varrho_n - 1]^+ \to 0 \quad \text{in } L^\infty\big(0,T; L^\alpha(\Omega)\big) \text{ for any } 1 \leqslant \alpha < \infty,$$

$$\varrho_n \to \varrho \quad \text{weakly star in } L^\infty\big(0,T; L^\alpha(\Omega)\big),\ 1 \leqslant \alpha < \infty,$$

where

$$0 \leqslant \varrho \leqslant 1.$$

Moreover, $\varrho_n^{\gamma_n}$ is bounded in $L^1((0,T)\times\Omega)$ and

$$\varrho_n^{\gamma_n} \to \mathcal{P} \quad \text{weakly star in } \mathcal{M}\big((0,T)\times\Omega\big).$$

If, in addition, $\varrho_n^0 \to \varrho^0$ strongly in $L^1(\Omega)$, then ϱ, $\mathbf{u}$, $\mathcal{P}$ solve the problem (8.14)–(8.17) *in $\mathcal{D}'((0,T)\times\Omega)$ where $\mathbf{u}$ is a weak limit of $\mathbf{u}_n$ in $L^2(0,T,W_0^{1,2}(\Omega))$.*

Here $\mathcal{M}$ denotes the space of Radon measures.

9. Other topics, directions, alternative models

9.1. *Models in one space dimension*

In the above analysis, we have systematicaly and deliberately avoided the case of one space dimension. Note that for *compressible* fluids such a situation can be physically relevant as well as interesting. From the mathematical point of view, these problems exhibit a rather different character due to the particularly simple topological structure of the underlying spatial domain.

The question of global existence is largely settled in the case of one space dimension. The basic result in this direction is that of Kazhikhov [56], a more extensive material can be found in the monograph of Antontsev et al. [4]. The discontinuous (weak) solutions were studied by Hoff [45], Serre [90,91] and Shelukhin [95]. The results are quite satisfactory with respect to the criteria of well-posedness discussed in Section 2. Jiang [54] proved global existence for the full system in one space dimension when the viscosity coefficients depend on the density. Probably the most general result as well as an extensive list of relevant literature is contained in the recent paper by Amosov [2].

There is a vast amount of literature concerning the qualitative properties of solutions. Straškraba [98,99], Straškraba and Valli [100] and Zlotnik [113] studied the long time-behaviour of global solutions in the barotropic case driven by a nonzero external force. Similar results for the full system were obtained in [35]. More information can be found in Amosov and Zlotnik [3], Hsiao and Luo [53], Matsumura and Yanagi [74] and many others. A complete list of references goes beyond the scope of the present paper.

9.2. *Multi-dimensional diffusion waves*

A more detailed description of the long-time behaviour for the barotropic case in several space dimensions was obtained by Hoff and Zumbrun [52].

Following their presentation we consider the system (5.3), (5.4) with $\mathbf{f}=0$ on the whole space $\Omega=R^3$. The initial data

$$\varrho(0)=\varrho_0, \qquad (\varrho\mathbf{u})(0)=\mathbf{q}_0 \tag{9.1}$$

are smooth and close to the constant state $\varrho^*=1$, $\mathbf{q}_0=0$.

Under these circumstances, the problem (5.3), (5.4) admits a global solution and the following theorem holds.

THEOREM 9.1. *Assume that the initial data satisfy*

$$\|\varrho_0 - 1\|_{L^1 \cap W^{1+d,2}(R^3)} + \|\mathbf{q}_0\|_{L^1 \cap W^{1+d,2}(R^3)} < \varepsilon,$$

where $\varepsilon > 0$ *is sufficiently small and* $d \geqslant 3$ *is an integer.*

Then the initial-value problem (5.3), (5.4), (9.1) *possesses a global solution* ϱ, $\mathbf{u}$ *satisfying*

$$\begin{aligned}&\left\|\partial_x^\alpha(\varrho(t) - 1)\right\|_{L^p(R^3)} + \left\|\partial_x^\alpha(\varrho\mathbf{u})(t)\right\|_{L^p(R^3)}\\ &\quad\leqslant c(d)\varepsilon \begin{cases}(1+t)^{-r_{\alpha,p}} & \text{for } 2 \leqslant p \leqslant \infty,\\ (1+t)^{-r_{\alpha,p}+1/p-1/2} & \text{if } 1 \leqslant p < 2\end{cases}\end{aligned}$$

for any multi-index $|\alpha| \leqslant (d-3)/2$ *where*

$$r_{\alpha,p} = |\alpha|/2 + 3/2(1 - 1/p).$$

Theorem 9.1 shows that perturbations of the constant state decay at the rate of a heat kernel for $p \geqslant 2$ but less rapidly if $p < 2$; in fact, the bound may even grow with time in the latter case.

A more detailed picture of the long-time behaviour in the L^p-norm for $p \geqslant 2$ is provided by the following result.

THEOREM 9.2. *Under the assumptions of Theorem* 9.1, *we have*

$$\begin{aligned}&\left\|\partial_x^\alpha(\varrho - 1)(t)\right\|_{L^p(R^3)} + \left\|\partial_x^\alpha\big((\varrho\mathbf{u})(t) - \mathcal{K}_\mu * [P\mathbf{q}_0]\big)\right\|_{L^p(R^3)}\\ &\quad\leqslant c(d)\varepsilon(1+t)^{-r_{\alpha,p}+1/p-1/2},\end{aligned}$$

$p \geqslant 2$, *where* P *is the projection on the space of divergence free functions and* $\mathcal{K}_\mu$ *is the standard heat kernel, i.e., the fundamental solution of the problem*

$$\partial_t v - \mu\Delta v = 0.$$

Thus the dynamics in L^p, $p > 2$ is dominated by a term with constant density and a non-constant divergence-free momentum field decaying at the rate of the heat kernel. In other words, for $p > 2$, all smooth, small amplitude solutions are asymptotically incompressible.

Finally, consider an auxiliary problem:

$$\begin{aligned}&\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = \frac{1}{2}(\lambda + 2\mu)\Delta\varrho,\\ &\partial_t(\varrho\mathbf{u}) + p'(1)\nabla\varrho = \mu\Delta(\varrho\mathbf{u}) + \frac{1}{2}\lambda\nabla\operatorname{div}(\varrho\mathbf{u}).\end{aligned} \tag{9.2}$$

Let us denote

$$U(t) = \big[\varrho(t), (\varrho\mathbf{u})(t)\big]$$

the solution of the linear problem (9.2) with the initial data

$$\varrho(0) = \varrho_0 - 1, \qquad (\varrho\mathbf{u})(0) = \mathbf{q}_0.$$

The long-time dynamics in L^p, $p < 2$, is described as follows.

THEOREM 9.3. *Under the hypotheses of Theorem* 9.1*, we have*

$$\begin{aligned}&\left\| \partial_x^\alpha \left(\left[\varrho(t) - 1, (\varrho\mathbf{u})(t) \right] - U(t) \right) \right\|_{L^p(R^3)} \\ &\quad \leqslant c(l, \sigma)\varepsilon(1+t)^{-r_{\alpha,p}+3/4(2/p-1)-1/2+\sigma}, \quad 1 \leqslant p < 2,\end{aligned}$$

for any positive σ.

All results in this part are taken over from [52].

9.3. *Energy decay of solutions on unbounded domains*

Various authors have considered the long-time behaviour of solutions on unbounded domains.

Following Kobayashi and Shibata [58] we consider the full system (2.1)–(2.3) on an exterior domain $\Omega \subset R^3$ where the pressure $p = p(\varrho, \theta)$ is given by a general constitutive law conform with the basic thermodynamical principles expressed in (1.4).

As for the boundary conditions, we take

$$\begin{aligned}&\mathbf{u}|_{\partial\Omega} = 0, \qquad \theta|_{\partial\Omega} = \theta_{\mathrm{b}}, \\ &\lim_{|x|\to\infty} \mathbf{u}(t,x) = 0, \qquad \lim_{|x|\to\infty} \theta(t,x) = \theta_{\mathrm{b}}.\end{aligned}$$

Assuming the initial data

$$\varrho(0) = \varrho_0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0$$

are closed to a constant state $[\bar{\varrho}, 0, \theta_{\mathrm{b}}]$, Kobayashi and Shibata [58, Theorem 2] show the following decay rates:

$$\begin{aligned}&\left\| \varrho(t) - \bar{\varrho} \right\|_{L^2(R^3)} + \left\| \mathbf{u}(t) \right\|_{L^2(R^3)} + \left\| \theta(t) - \theta_{\mathrm{b}} \right\|_{L^2(R^3)} \leqslant ct^{-3/4}, \\ &\left\| \varrho(t) - \bar{\varrho} \right\|_{L^\infty(R^3)} + \left\| \mathbf{u}(t) \right\|_{L^\infty(R^3)} + \left\| \theta(t) - \theta_{\mathrm{b}} \right\|_{L^\infty(R^3)} \leqslant ct^{-5/4}.\end{aligned}$$

Related results were obtained by Deckelnick [17], Kobayashi [57], Padula [87] and many others.

9.4. *Alternative models*

Up to now, we have considered only Newtonian fluids where the viscous stress tensor Σ was a linear function of the velocity gradient $\nabla\mathbf{u}$. However, some experimental results show that in nature there exist stronger dissipative mechanisms not captured by the classical Stokes law. Let us shortly discuss this interesting and rapidly developing area of modern mathematical physics which gives an alterantive and, given the enormous amount of open problems in the classical theory, mathematically attractive way to describe the fluid motion.

In the linear theory of *multipolar fluids*, the constitutive laws, in particular, the viscous stress tensor Σ depend not only on the first spatial gradients of the velocity field $\mathbf{u}$ but also on the higher order gradients up to order $2k-1$ for the so-called k-polar fluids.

In the work of Nečas and Šilhavý [82], an axiomatic theory of viscous multipolar fluids was developed in the framework of the theory of elastic non-viscous multipolar materials due to Green and Rivlin [43]. Accordingly, the viscous stress tensor Σ takes a general form:

$$\Sigma = \sum_{j=0}^{k-1}(-1)^j\big(\mu_j\Delta^j\big(\nabla\mathbf{u}+(\nabla\mathbf{u})^t\big)+\lambda_j\Delta^j\operatorname{div}\mathbf{u}\,\mathrm{Id}\big)$$
$$+\,\omega\big(\nabla\mathbf{u}+(\nabla\mathbf{u})^t\big)+\beta\operatorname{div}\mathbf{u}\,\mathrm{Id}, \tag{9.3}$$

where, in the nonlinear component,

$$\beta=\beta\big(|\nabla\mathbf{u}|,\operatorname{div}\mathbf{u},\det(\nabla\mathbf{u})\big),\qquad \omega=\omega\big(|\nabla\mathbf{u}|,\operatorname{div}\mathbf{u},\det(\nabla\mathbf{u})\big).$$

The existence of the so-called measure-valued solutions of the initial value problem for isothermal flows, i.e., for the system (2.1), (2.2) with Σ given as in (9.3) and the pressure satisfying $p=r\theta_0\varrho$, was proved by Matušů-Nečasová and Novotný [75].

The weak solutions for linear multipolar fluids were obtained in a series of papers by Nečas et al. [81,80].

Recently, new results concerning the so-called *power-law fluids*, i.e., when $k=1$ in (9.3), were shown by Mamontov [69,70].

10. Conclusion

Despite the enormous progress during the last two decades, we still seem to be very far from a satisfactory rigorous mathematical theory of viscous compressible and/or heat conducting fluids. There are good an bad news according to the degree of complexity of the problems considered but we still wait, for instance, for a large data existence result for, say, the isothermal flow in two and three space dimensions. On the point of conclusion, let us discuss shortly the major mathematcal difficulties presently encountered.

10.1. *Local existence and uniqueness, small data results*

As we have seen in Section 2, the initial value problem for the full system (1.1)–(1.3) complemented by physically relevant constitutive relations admits a unique global in time classical solution. From the mathematical point of view, this is nothing less or more than to say that the *linearized system* is well-posed. This fact seems to be the primary criterion of applicability of *any* mathematical model. Indeed there is only a little to say should the linearized problem be ill-posed. However, there still can remain an essential gap between “linear” and “nonlinear” provided there is no dissipative meachanism present as it is the case for nonlinear hyperbolic systems. The possibility to construct classical though only “small” solutions reveals the *dissipative* character of the problem, namely, the effect of the diffusion terms present in the parabolic equations (1.2), (1.3).

Another aspect of dissipativity is the existence of bounded absorbing sets discussed in Section 7.1 and the existence of global attractors mentioned in Section 7.5. Although we still do not know if the attractor has a finite fractal dimension, there are strong indications (cf. Hoff and Ziane [50]) it might be the case.

10.2. *Density estimates*

Unlike (1.2), (1.3), the continuity equation (1.1) governing the time evolution of the density is hyperbolic and linear with respect to ϱ. As a consequence, one cannot expect any smoothing effect as for parabolic problems or compactification phenomena as it is the case for genuinely nonlinear hyperbolic equations. We have made it clear several times in this paper that the major obstacle to develop a rigorous large data theory for our problem is the lack of *a priori estimates* on the density ϱ.

The density being a non-negative function there are two aspects of the problem – boundedness from below away from zero and uniform upper bounds. Let us remark that the system (1.1)–(1.3) and, in particular, the constitutive relations for Newtonian fluids hold for nondilute fluids with no vacuum zones.

Let us review the results of Desjardins [18] illuminating the role of upper bounds on ϱ in the well-posedness problem. Consider the isentropic model represented by the system (5.3), (5.4) where the pressure p satisfies (5.2) with $\gamma > 1$. For simplicity, we consider the case of spatially periodic boundary conditions in two space dimensions. The following result is proved by Desjardins [18, Theorem 2].

THEOREM 10.1. *Consider the system* (5.3), (5.4) *where*

$$p(\varrho) = a\varrho^{\gamma}, \quad a > 0, \ \gamma > 1,$$

in two space dimensions and with spatially periodic data

$$\varrho(0) \geqslant 0 \in L^{\infty}_{\mathrm{per}}(R^2), \qquad \mathbf{u}(0) \in W^{1,2}_{\mathrm{per}}(R^2), \qquad \mathbf{f} \equiv 0.$$

Then there exists $T_0 > 0$ and a weak solution ϱ, $\mathbf{u}$ of the problem such that for all $T < T_0$

$$\varrho \in L^{\infty}\big(0, T; L^{\infty}_{\mathrm{per}}\big(R^2\big)\big)$$

and

$$\sqrt{\varrho}\partial_t \mathbf{u} \in L^2\big(0, T; L^2_{\mathrm{per}}\big(R^2\big)\big),$$

$$p - (\lambda + 2\mu)\operatorname{div}\mathbf{u} \in L^2\big(0, T; W^{1,2}_{\mathrm{per}}\big(R^2\big)\big),$$

$$\nabla u \in L^{\infty}\big(0, T; L^2_{\mathrm{per}}\big(R^2\big)\big),$$

$$\mathcal{P}\mathbf{u} \in L^2\big(0, T; W^{2,2}_{\mathrm{per}}\big(R^2\big)\big),$$

where $\mathcal{P}$ denotes the projection on the space of divergence free functions. Moreover, the regularity properties stated above hold as long as

$$\sup_{t\in[0,T]} \big\|\varrho(t)\big\|_{L^{\infty}_{\mathrm{per}}(R^2)} < \infty.$$

Desjardins [18] proved also that the weak solutions constructed in Theorem 10.1 enjoy the weak-strong uniqueness property well-known from the theory of incompressible flows. Specifically, the above weak solution coincides with a strong one as long as the latter exists (cf. [18, Theorem 3]).

The lower bounds on the density represent an equally delicate issue. As we have seen in Section 7.3, one cannot avoid vacuum states provided we accept the isentropic model as an adequate description for the *long time* behaviour of solutions. On the other hand, the density should remain strictly positive for any *finite time t* provided its initial distribution enjoys this property. Unfortunately, however, this is not known in the class of weak solutions provided $N \geqslant 2$. To reveal the pathological character of the problem when vacua are present, we follow Liu et al. [66] and consider the isentropic model in one space dimension where the initial distribution of the density is given as

$$\varrho(0) = \varrho_0(x - 2r) + \varrho_0(x + 2r),$$

where ϱ_0 is a compactly supported smooth function with support contained in the ball $\{|x| < r\}$. Should the model correspond to physical intuition, one would expect, at least on a short time interval, the solution to be given as

$$\varrho(t, x) = \tilde{\varrho}(t, x - 2r) + \tilde{\varrho}(t, x + 2r),$$
$$u(t, x) = \tilde{u}(t, x - 2r) + \tilde{u}(t, x + 2r),$$

where $\tilde{\varrho}$, $\tilde{u}$ solve the problem for the initial data $\tilde{\varrho}(0) = \varrho_0$. However, as shown in [66], this is not the case. Of course, this apparent difficulty is due to the discrepancy between the *finite speed of propagation* property which holds for the hyperbolic equation (5.3) and the *instantaneous propagation* due to the diffusion character of (5.4) (for other unusual features of the problem we refer also to Hoff and Serre [48]). In fact, one should consider the viscosity coefficients μ and λ depending on the density ϱ in this case (see Jiang [54]).

The formation or rather non-formation of vacua has been studied in a recent paper by Hoff and Smoller [49]. They prove that the weak solutions of the Navier–Stokes equations for compressible fluid flows in *one space dimension* do not exhibit vacuum states in a finite time provided that no vacuum is present initially under fairly general conditions on the data. Unfortunately, however, such a result is not known in higher space dimensions even when the data exhibit some sort of symmetry, say, they are radially symmetric with respect to origin.

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